Integrated Math 3
Module 1
Functions and Their Inverses

Adapted from

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Module 1 Overview

Prerequisite Concepts and Skills:

- Domain
- Range
- Transformations of functions
- Definition of exponential functions as equal differences over equal intervals (Integrated Math 1, Module 3)
- Finding an inverse of a function
- Determining if two functions are inverses
- Writing linear, exponential, and quadratic functions from contexts

Summary of the Concepts & Skills in Module 1:

- Inverses of a function
- Restricted domain
- Inverse function notation
- Definition of a logarithm
- Symmetry of inverses

Content Standards and Standards for Mathematical Practice Covered:

- Content Standards: F.BF.1, F.BF.4a, F.BF.4b, F.BF.4c, F.BF.4d, F.BF.5
- Standards for Mathematical Practice:
  1. Make sense of problems and persevere in solving them.
  2. Reason abstractly and quantitatively.
  3. Construct viable arguments and critique the reasoning of others.
  4. Model with mathematics.
  5. Use appropriate tools strategically.
  6. Attend to precision.
  7. Look for and make use of structure.
  8. Look for and express regularity in repeated reasoning.

Module 1 Vocabulary:

- Inverse
- Restricted domain
- Logarithm
- Vertical asymptote
- One-to-one
- Invertible

Concepts Used In the Next Module:

- Inverses
- Logarithms
- Exponential functions
- Features of functions
- Exponential properties
- Transformations of functions

Module 1 – Functions and Their Inverses

1.1 *Develops the concept of inverse functions in a linear modeling context using tables, graphs, and equations.* **(F.BF.1, F.BF.4, F.BF.4a)**
Warm Up: Pet Food
Classroom Task: Brutus Bites Back – A Develop Understanding Task
Ready, Set, Go Homework: Functions and Their Inverses 1.1

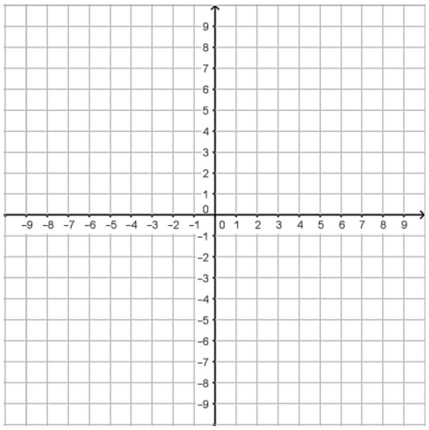
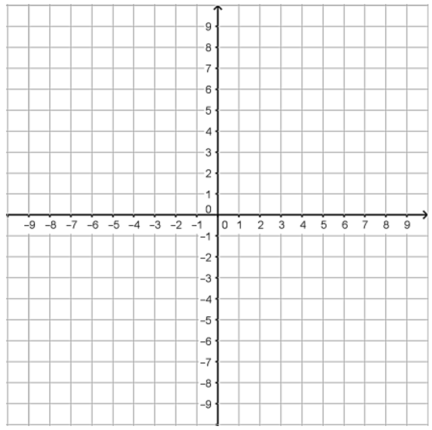
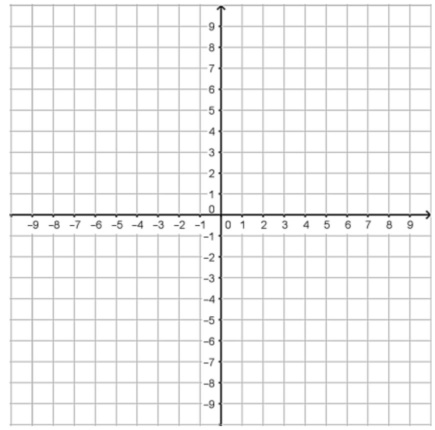
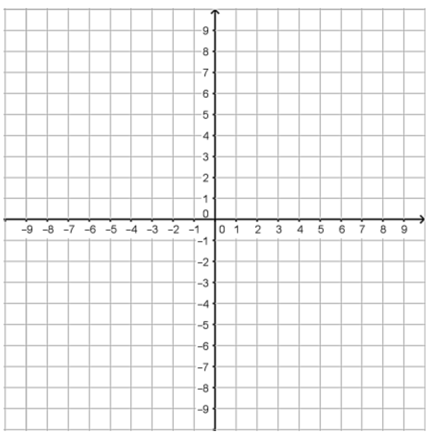
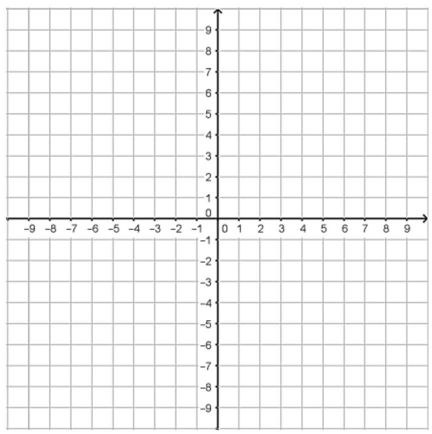
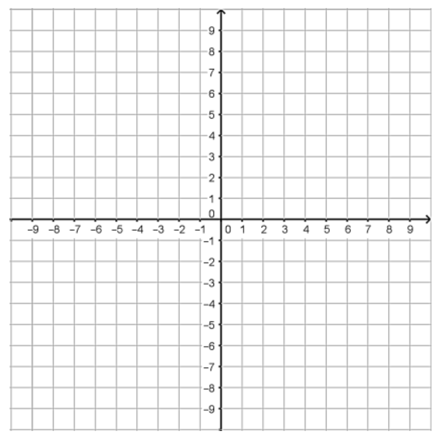
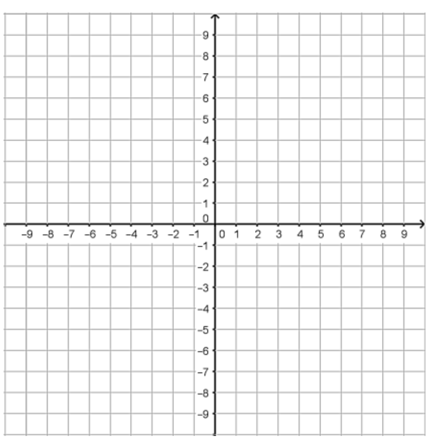
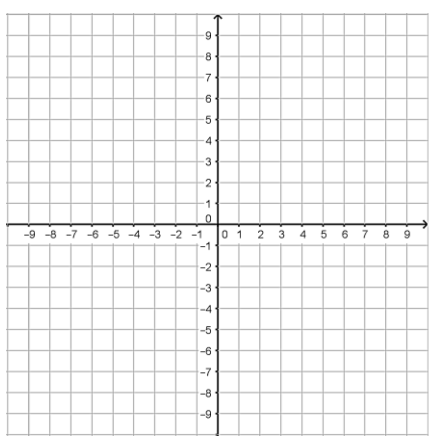
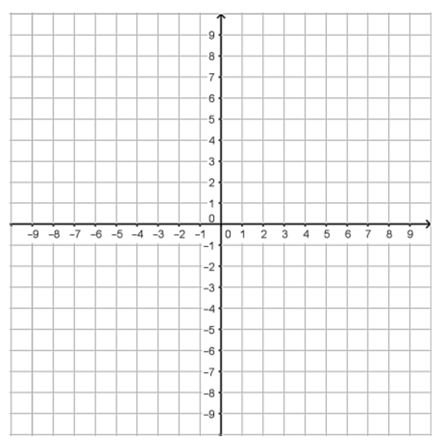
1.2 *Extends the concepts of inverse functions in a quadratic modeling context with a focus on domain and range and whether a function is invertible in a given domain.* **(F.BF.1, F.BF.4, F.BF.4c, F.BF.4d)**
Warm Up: Function or Not a Function
Classroom Task: Flipping Ferraris – A Solidify Understanding Task
Ready, Set, Go Homework: Functions and Their Inverses 1.2

1.3 *Solidifies the concepts of inverse function in an exponential modeling context and surfaces ideas about logarithms.* **(F.BF.1, F.BF.4, F.BF.4c, F.BF.4d, F.BF.5)**
Warm Up: Begin task
Classroom Task: Tracking the Tortoise – A Solidify Understanding Task
Ready, Set, Go Homework: Functions and Their Inverses 1.3

1.4 *Uses function machines to model functions and their inverses. Focus on finding inverse functions and verifying that two functions are inverses.* **(F.BF.4, F.BF.4a, F.BF.4b)**
Warm Up: Inverse Functions
Classroom Task: Pulling a Rabbit Out of a Hat – A Solidify Understanding Task
Ready, Set, Go Homework: Functions and Their Inverses 1.4

1.5 *Uses tables, graphs, equations, and written descriptions of functions to match functions and their inverses together and to verify the inverse relationship between two functions.* **(F.BF.4a, F.BF.4b, F.BF.4c, F.BF.4d)**
Warm Up: Multiple Representations of Inverse
Classroom Task: Inverse Universe – A Practice Understanding Task
Ready, Set, Go Homework: Functions and Their Inverses 1.5

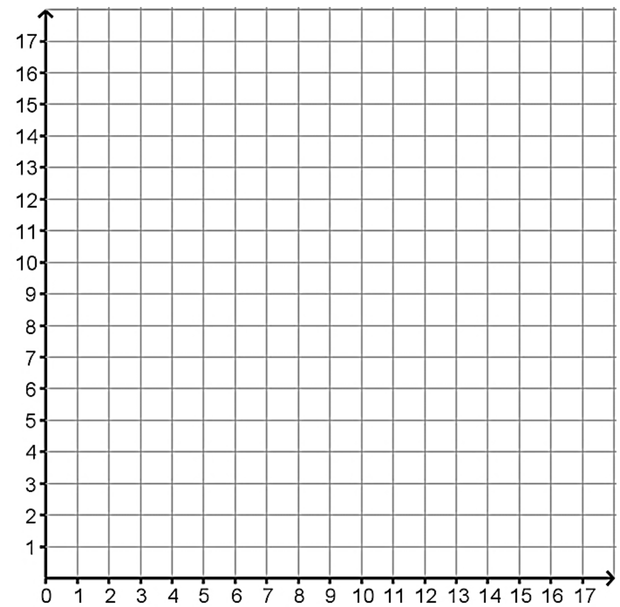
Math 1 and Math 2
Parent Functions and Conic Sections

$y = c$ 	$y = x$ 	$y = 2^x$ 
$y = x^2$ 	$y = \|x\|$ 	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ 
$x^2 + y^2 = r^2$ 	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ 	$x = y^2$ 

1.1 Warm Up
Pet Food

Carlos and Clarita need to provide nutritious food to the cats they are taking care of. They have spoken to their uncle, who is a veterinarian, about this. He suggests that the cats get 30 grams of protein per day. The two food brands they are giving to the cats (Tabitha Tidbits and Figaro Flakes) contain two different amounts of protein. Tabitha Tidbits contain 2 grams per serving, whereas Figaro Flakes contain 6 grams per serving.

Model the combinations of food servings possible in the space below (be sure to use labels):



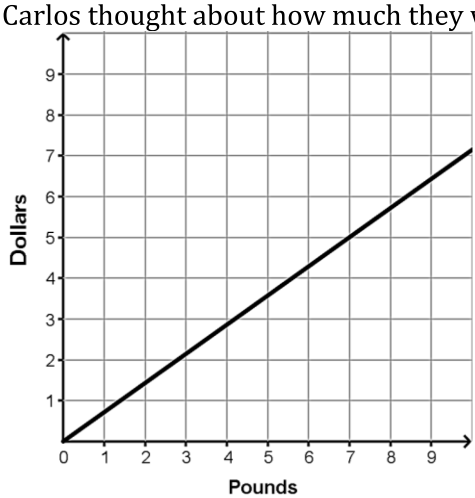
Equation:

1.1 Brutus Bites Back
A Develop Understanding Task

Remember Carlos and Clarita from Math 1? A couple of years ago, they started earning money by taking care of pets while their owners were away. Due to their amazing mathematical analysis and their loving care of the cats and dogs, Carlos and Clarita have made their business very successful. To keep the hungry dogs fed, they must regularly buy Brutus Bites, the favorite food of all the dogs.

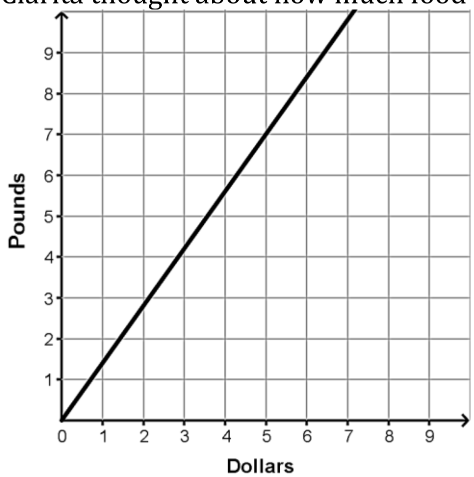
Carlos and Clarita have been searching for a new dog food supplier and have identified two possibilities. The first company which is located in their hometown, Canine Catering Company, sells 7 pounds of food for $5.

Carlos thought about how much they would pay for a given amount of food and drew this graph:



1. Using function notation, write the equation of the function that Carlos graphed.

Clarita thought about how much food they could buy for a given amount of money and drew this graph:



2. Using function notation, write the equation of the function that Clarita graphed.

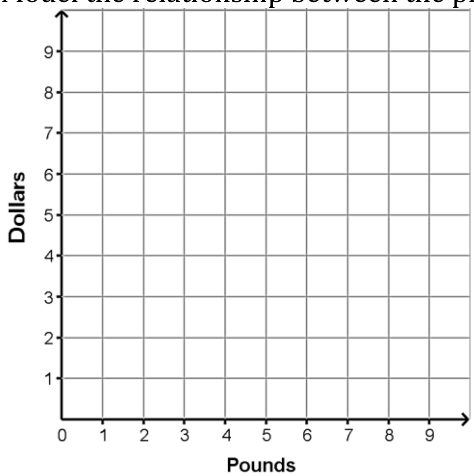
3. Write a question that would be most easily answered by Carlos' graph. Write a question that would be most easily answered by Clarita's graph. What is the difference between the two questions?

4. What is the relationship between the two functions? How do you know?

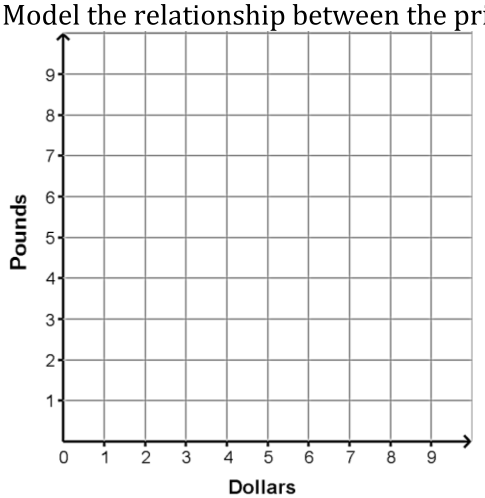
5. Use function notation to write the relationship between the functions.

Looking online, Carlos found a company that will sell 8 pounds of Brutus Bites for $6 plus a flat $5 shipping charge for each order. The company advertises that they will sell any amount of food at the same price per pound.

6. Model the relationship between the price and the amount of food using Carlos' approach.



7. Model the relationship between the price and the amount of food using Clarita's approach.



8. What is the relationship between these two functions? How do you know?

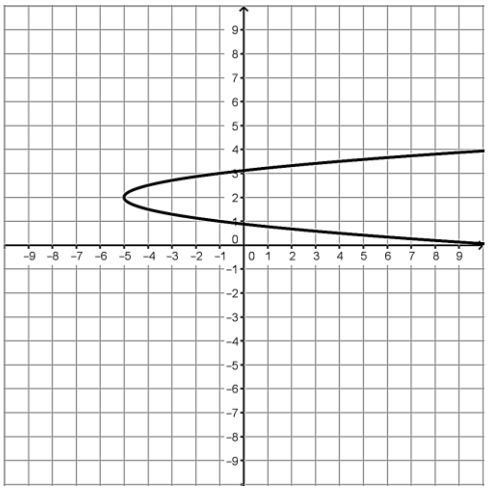
9. Use function notation to write the relationship between the functions.

10. Which company should Clarita and Carlos buy their Brutus Bites from? Why?

1.2 Warm Up
Function or Not a Function

Determine if each relation is a function or not. Jusify your answers.

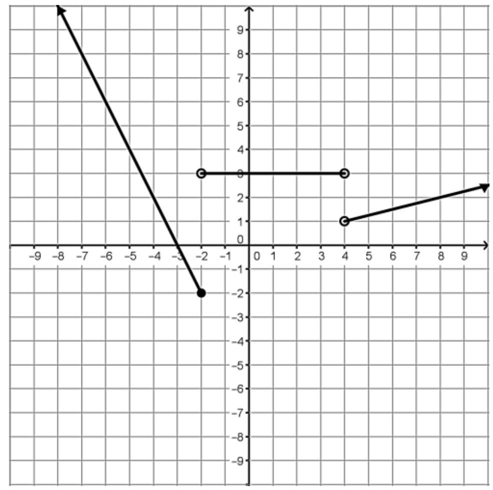
1.



2.

Input	Output
12	5
7	9
4	5
-3	-2
9	-1

3.



Solve each each below. Simplify your answers as much as possible.

4. $\sqrt[3]{2x-1}+6=8$

5. $2\sqrt{3x+5}-1=7$

6. $3(x+4)^2+8=35$

7. $\sqrt[3]{567}=\sqrt[3]{x^2-9}$

8. $27^x=81$

9. $7^x=\frac{1}{343}$

1.2 Flipping Ferraris
A Solidify Understanding Task

When people first learn to drive, they are often told that the faster they are driving, the longer it will take to stop. So, when you drive on the freeway, you should leave more space between your car and the car in front of you than you would if you were driving slowly through a neighborhood. Have you ever wondered about the relationship between how fast you are driving and how far you travel before you stop, after hitting the brakes?

1. What factors do you think might make a difference in how far a car travels after hitting the brakes?

There has actually been quite a bit of experimental work done to be able to mathematically model the relationship between the speed of a car and the braking distance (how far the car goes prior to stopping after the driver hits the brakes).

2. Imagine your dream car. Maybe it is a Ferrari 550 Maranello, a super-fast Italian car. Experiments have shown that on smooth, dry roads, the relationship between the braking distance (d) and speed (s) is given by $d(s) = 0.03s^2$. Speed is given in miles/hour and the distance is in feet.

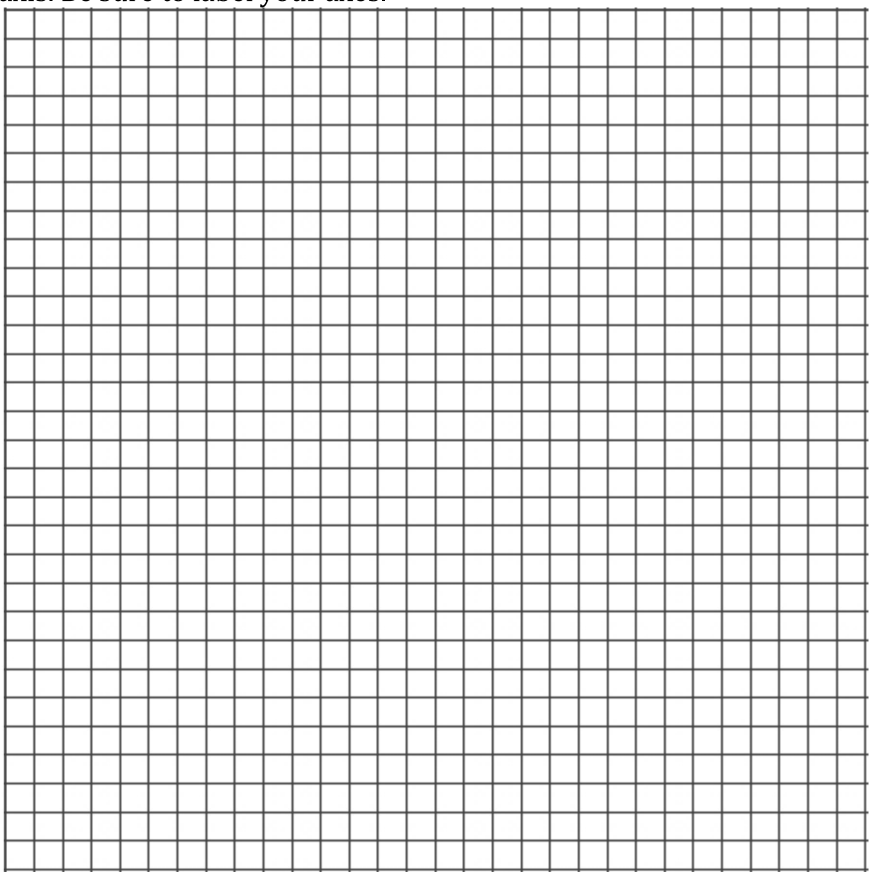
   a. How many feet should you leave between you and the car in front of you if you are driving the Ferrari at 55 mi/hr?

   b. What distance should you keep between you and the car in front of you if you are driving at 100 mi/hr?

   c. If an average car is about 16 feet long, about how many car lengths should you have between you and that car in front of you if you are driving 100 mi/hr?

   d. It makes sense to a lot of people that if the car is moving at some speed and then goes twice as fast, the braking distance will be twice as far. Is that true? Explain why or why not.

3. Graph the relationship between braking distance $d(s)$, and speed (s), below. Use an equal scale on each axis. Be sure to label your axes.



4. Describe all the mathematical features of the relationship between braking distance and speed for the Ferrari modeled by $d(s) = 0.03s^2$. Be sure to discuss features such as domain, range, intercepts, maximum/minimum values, intervals of increase/decrease/constant.

5. What if the driver of the Ferrari 550 was cruising along and suddenly hit the brakes to stop because she saw a cat in the road? She skidded to a stop and fortunately, missed the cat. When the driver got out of the car, she measured her car's skid marks. She found that her braking distance was 31 ft.

   a. How fast was she going when she hit the brakes?

   b. If she didn't see the cat until she was 15 feet away, what is the fastest speed she could travel before hitting the brakes in order to avoid hitting the cat?

6. Part of a police officer's job is to investigate traffic accidents to determine what caused the accident and which driver was at fault. Police officers measure the braking distance using skid marks and calculate speeds using the mathematical relationships like we just examined. Police officers often use different formulas to account for various factors such as road conditions and weather. Let's go back to the Ferrari on a smooth, dry road since we know the relationship. Create a table that shows the speed the car was traveling based upon the braking distance.

7. Write an equation of the function $s(d)$ that gives the speed the car was traveling for a given braking distance.

8. Graph the function $s(d)$ and describe its features. Use the same scale as you did in question 3. Be sure to label your axes.

9. What do you notice about the graph of $s(d)$ compared to the graph of $d(s)$? What is the relationship between the functions $d(s)$ and $s(d)$?

10. Consider the function $d(s) = 0.03s^2$ over the domain of all real numbers. How does this graph, with an expanded domain, change from the graph of $d(s)$ in question #3?

11. How does changing the domain of $d(s)$ change the graph of the inverse of $d(s)$?

12. Is the inverse of $d(s)$ a function? Justify your answer.

13. How are functions and their inverses related to each other? Be sure to discuss relationships between domain/range, graphs, and table of values.

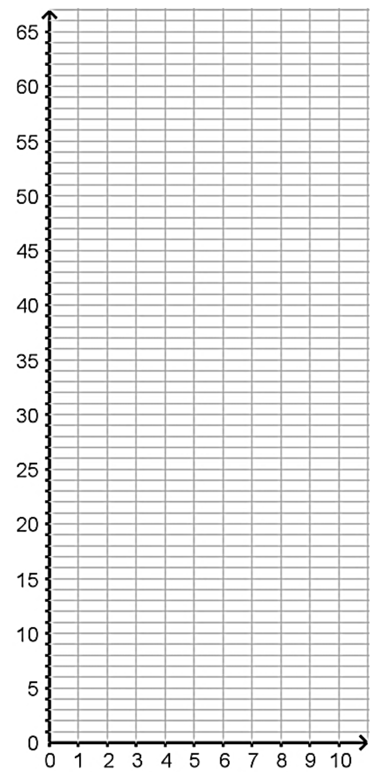
1.3 Tracking the Tortoise
A Solidify Understanding Task

You may remember a task from last year about the famous race between the tortoise and the hare. In the children's story of the tortoise and the hare, the hare mocks the tortoise for being slow. The tortoise replies, "Slow and steady wins the race." The hare says, "We'll just see about that," and challenges the tortoise to a race.

In the task, *Tortoise and the Hare,* we modeled the distance from the starting line that both the tortoise and the hare travelled during the race. Today we will consider only the journey of the tortoise in the race.

Because the hare is so confident that he can beat the tortoise, he gives the tortoise a 1 meter head start. The tortoise's distance from the starting line, including the head start, is given by the function: $d(t) = 2^t$ (d is in meters and t is in seconds).

The tortoise family decides to watch the race from the side lines so that they can see their darling tortoise sister, Shellie, prove the value of persistence.



1. How far away from the starting line must the family be, to be located in the right place for Shellie to run by 10 seconds after the beginning of the race? After 20 seconds?

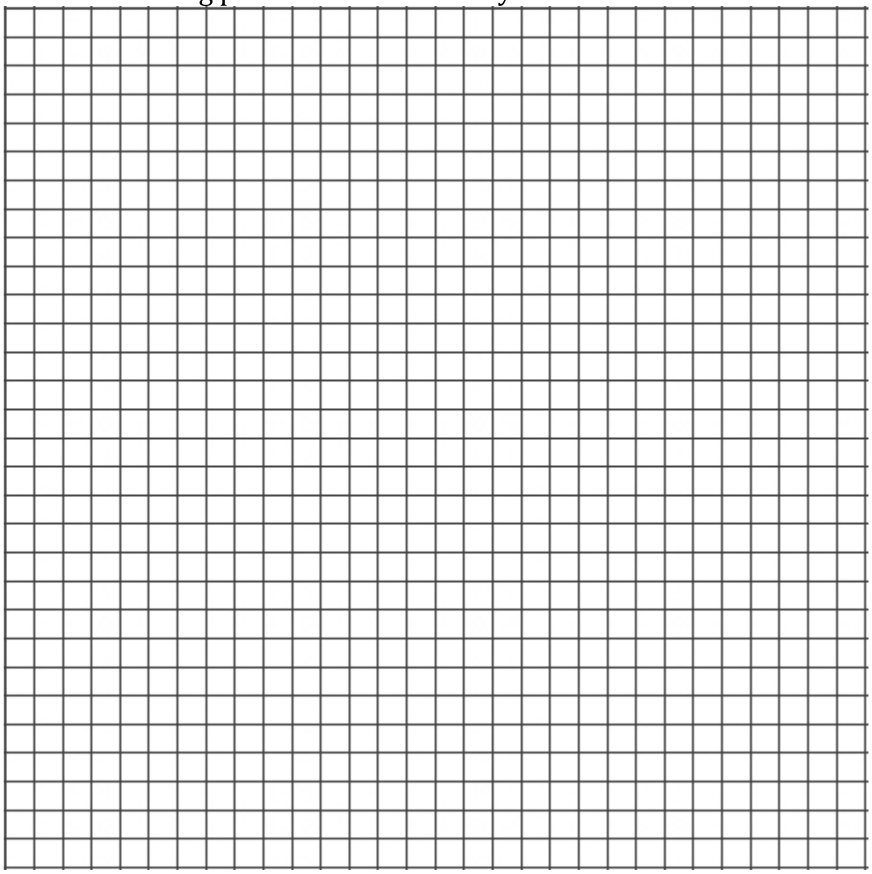
2. a. Describe the graph of $d(t)$, Shellie's distance at time t. What are the important features of $d(t)$?

   b. Graph $d(t)$ at right.

3. If the tortoise family plans to watch the race at 64 meters away from Shellie's starting point, how long will they have to wait to see Shellie run past?

4. How long must they wait to see Shellie run by if they stand 1024 meters away from her starting point?

5. Draw a graph that shows how long the tortoise family will wait to see Shellie run by at a given location from her starting point. Be sure to label your axes.



6. How long must the family wait to see Shellie run by if they stand 220 meters away from her starting point? Use your graph to estimate this value if necessary.

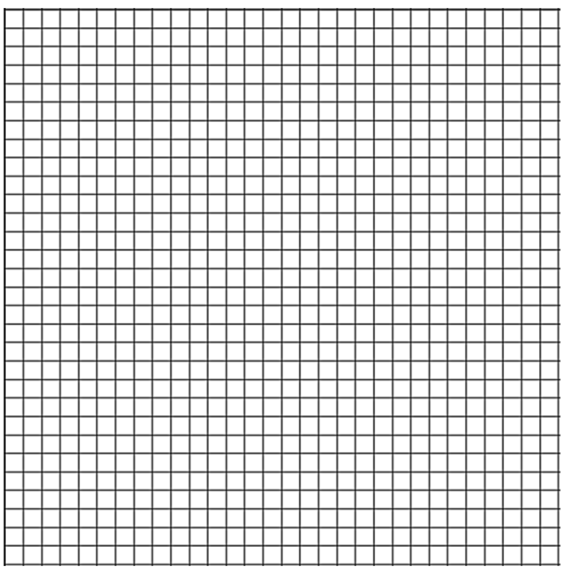
7. What is the relationship between $d(t)$ and the graph that you have just drawn? How did you use $d(t)$ to draw the graph in #5?

8. Consider the function $f(x) = 2^x$.
   a. Complete the table of values for $f(x)$ and use it to complete a table of values for the inverse, $f^{-1}(x)$.

x	$f(x)$
-3	
-2	
-1	
0	
1	
2	
3	

x	$f^{-1}(x)$

   b. Graph $f(x)$ and $f^{-1}(x)$ on the grid below. Also, graph the line $y = x$. Be sure to label the axes.



   c. Why is the inverse of $f(x)$ a function? That is, why is $f(x)$ **invertible**? Explain.

   d. Complete the table below with details of $f(x)$ and $f^{-1}(x)$.

	$f(x)$	$f^{-1}(x)$
Domain		
Range		
x-intercept(s)		
y-intercept		
Describe any asymptotes		

e. How does the domain and range of $f(x)$ compare to $f^{-1}(x)$? What do you notice about the intercepts? The asymptotes? Explain.

9. If $f(3) = 8$, what is $f^{-1}(8)$? How do you know?

10. If $f\left(\frac{1}{2}\right) = 1.414$, what is $f^{-1}(1.414)$? How do you know?

11. If $f(a) = b$ what is $f^{-1}(b)$? Will your answer change if $f(x)$ is a different function? Explain.

12. Mathematicians have come up with a way to express $f^{-1}(x)$. If $y = 2^x$, then the inverse is written $y = \log_2 x$. In the notation, $y = \log_2 x$, 2 is called the **base** and x is called the **argument** of the logarithm.

> Using the relationship between $f(x)$ and $f^{-1}(x)$ you have explored in this task, we can define **logarithms** as:
>
> *For all positive numbers b, where $b \neq 1$, and all positive numbers x,* $\boldsymbol{y = \log_b x}$ *is equivalent to* $\boldsymbol{x = b^y}$.
> *(Note the **base** of the exponent and the **base** of the logarithm are both **b**.)*

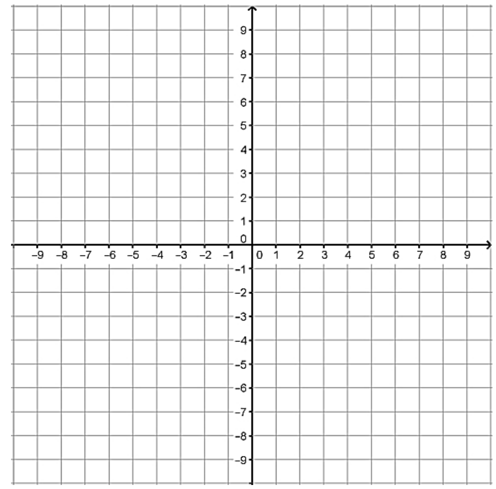
<u>Example:</u> If $f(5) = 32$, then $f^{-1}(32) = 5$.
If $32 = 2^5$, then $5 = \log_2 32$.

Use this new notation to express your answers from questions 9-11 for $f(x) = 2^x$ (reminder, $y = f(x)$).

1.4 Warm Up
Inverse Functions

1. Consider the function $f(x) = x^2$. Graph $f(x)$ and its inverse.

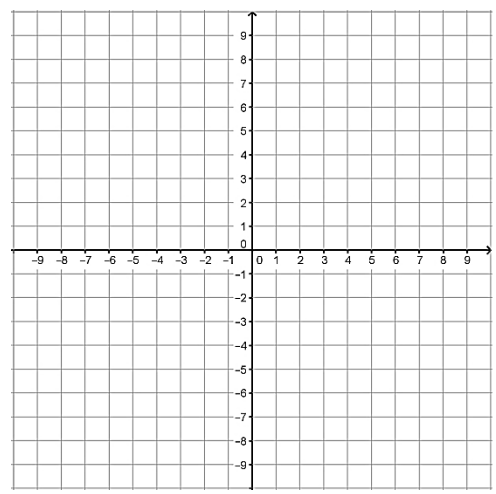
2. a. Is $f(x)$ invertible? In other words, is the inverse of $f(x)$ a function?



   b. Express the inverse of $f(x) = x^2$ using an equation.

   c. What restrictions must be made on the domain of $f(x)$ in order to be **invertible**? In other words, what restrictions must be made on the domain of $f(x)$ to ensure that its inverse is also a function? List all possible restrictions.

3. a. Find the inverse of the function: $g(x) = 2(x+4)^2 - 6$



   b. What restrictions must be made on the domain of $g(x)$ in order to be invertible? List all possible restrictions.

   c. Graph $g(x)$ with its restriction as well as the function $g^{-1}(x)$.

4. The vertical line test is used to determine if a graph represents a function. Explain how a horizontal line could be used to determine if the inverse of a graph of a function is also a function?

1.4 Pulling a Rabbit Out of a Hat
A Solidify Understanding Task

I have a magic trick for you:

- Pick a number, any number.
- Add 6
- Multiply by the result by 2
- Subtract 12
- Divide by 2
- The answer is the number you started with!

People are often mystified by such tricks but those of us who have studied inverse operations and inverse functions can easily figure out how they work and even create our own number tricks. Let's get started by figuring out how inverse functions work together.

For each of the following function machines, decide what function can be used to make the output the same as the input number. Describe the operation(s) in words needed to produce equivalent inputs and outputs and then write the inverse function symbolically. You may wish to choose other input values to verify your inverse function works (such as $x = -10, 0, \frac{1}{2}, 5$).

Here's a couple of examples:

Input $\qquad$ Output

$x = 7 \longrightarrow 3 \cdot 7 = 21 \longrightarrow 7$

$f(x) = 3x$ $\qquad$ $\boxed{f^{-1}(x) = \frac{x}{3}}$

In words: ***Take the output of 21 and divide it by 3 to get the original input of 7.***

Input $\qquad$ Output

$x = 7 \longrightarrow 2 \cdot 7 - 5 = 9 \longrightarrow 7$

$f(x) = 2x - 5$ $\qquad$ $\boxed{f^{-1}(x) = \frac{x+5}{2}}$

In words: ***Take the output of 9, add 5, then divide by 2 to get the original input of 7.***

1.

Input $\qquad$ Output

$x = 7 \longrightarrow \frac{7+5}{3} = 4 \longrightarrow 7$

$f(x) = \frac{x+5}{3}$ $\qquad$ $\boxed{f^{-1}(x) = \quad}$

In words:

2.

Input | Output

$$x = 7 \xrightarrow{f(x) = x^2} 7^2 = 49 \xrightarrow{\boxed{f^{-1}(x) = \quad}} 7$$

In words:

3.

Input | Output

$$x = 7 \xrightarrow{f(x) = 2^x} 2^7 = 128 \xrightarrow{\boxed{f^{-1}(x) = \quad}} 7$$

In words:

4.

Input | Output

$$x = 7 \xrightarrow{f(x) = (x-3)^2} (7-3)^2 = 16 \xrightarrow{\boxed{f^{-1}(x) = \quad}} 7$$

In words:

5.

Input $\qquad\qquad\qquad\qquad$ Output

$$x = 7 \xrightarrow{f(x) = 4 - \sqrt{x}} 4 - \sqrt{7} \xrightarrow{\boxed{f^{-1}(x) = \qquad}} 7$$

In words:

6.

Input $\qquad\qquad\qquad\qquad$ Output

$$x = 7 \xrightarrow{f(x) = 2^x - 10} 2^7 - 10 = 118 \xrightarrow{\boxed{f^{-1}(x) = \qquad}} 7$$

In words:

7. The functions $f(x) = 2x - 4$ and $g(x) = \frac{1}{2}x + 2$ are inverses. What is the composition $f\big(g(x)\big)$? What is the composition $g\big(f(x)\big)$?

8. What can be said about the domain and range of the function and its inverse from question 5?

9. Each of these problems began with an input of $x = 7$. What is the difference between the x used as the input for $f(x)$ and the x used as the input for $f^{-1}(x)$?

10. In #4, could any value of x be used in $f(x)$ and still give the same output from $f^{-1}(x)$? Explain. What about #5?

11. Based on your work in this task and other tasks in this module, what relationships exist between functions and their inverses?

1.5 Warm Up
Multiple Representations of Inverses

1. Complete each representation of the function and its inverse. Find the domain and range for both functions. Then use composition of functions $\left(f\left(f^{-1}(x)\right) \text{ and } f^{-1}\left(f(x)\right)\right)$ to verify that they are inverses.

$f(x) = \sqrt{x-5}$

x	$f(x)$

Domain of $f(x)$:

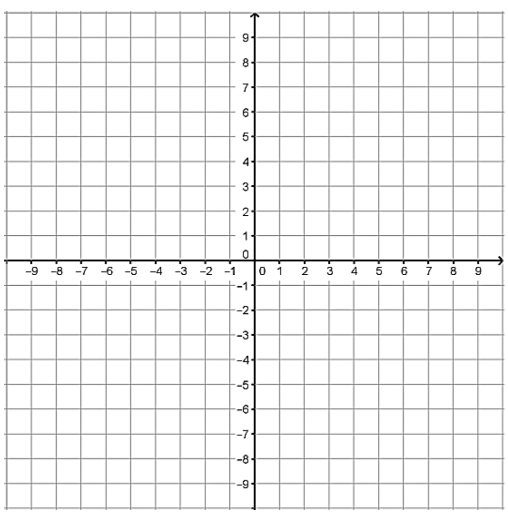
Range of $f(x)$:

$f^{-1}(x) =$

x	$f^{-1}(x)$

Domain of $f^{-1}(x)$:

Range of $f^{-1}(x)$:



Composition of functions:

$f\left(f^{-1}(x)\right) =$

$f^{-1}\left(f(x)\right) =$

2. Find $f^{-1}(x)$ for $f(x) = -\sqrt{x-5}$. Be sure to state the restricted domain of $f(x)$ (if any is needed).

3. Find $f^{-1}(x)$ for $f(x) = x^2 + 4, x \geq 0$. Be sure to state the restricted domain of $f(x)$ (if any is needed).

1.5 Inverse Universe
A Practice Understanding Task

You and your partner have each been given a different set of cards. The instructions are:

1. Select a card and show it to your partner.
2. Work together to find a card in your partner's set of cards that represents the inverse of the function represented on your card.
3. Record the cards you selected and the reason that you know that they are inverses in the space below.
4. Repeat the process until all of the cards are paired up.

*For this task only, assume that all tables represent points on a continuous function.

Pair 1:	Justification of inverse relationship:
Pair 2:	Justification of inverse relationship:
Pair 3:	Justification of inverse relationship:
Pair 4:	Justification of inverse relationship:
Pair 5:	Justification of inverse relationship:

Pair 6:	Justification of inverse relationship:
Pair 7:	Justification of inverse relationship:
Pair 8:	Justification of inverse relationship:
Pair 9:	Justification of inverse relationship:
Pair 10:	Justification of inverse relationship:

In the space below, address the following:

What does it mean to "justify an inverse relationship?" In other words, how can you tell if two relationships are inverses? Be sure you explain how using tables, graphs, and equations can be used to "justify an inverse."

Integrated Math 3
Module 1
Functions and Their Inverses
Ready, Set, Go! Homework

Adapted from

The Mathematics Vision Project:
Scott Hendrickson, Joleigh Honey, Barbara Kuehl,
Travis Lemon, Janet Sutorius

Ready, Set, Go!

Ready

Topic: Inverse operations

Inverse operations *"undo"* each other. For instance, addition and subtraction are inverse operations. So are multiplication and division. In mathematics, it is often convenient to *undo* several operations in order to solve for a variable.

In the middle column below, describe what happens to a number when plugged in for the variable *x* on the left side of the equation. In the right column below, describe the steps needed to solve for *x* in each equation. The first one has been done for you as an example.

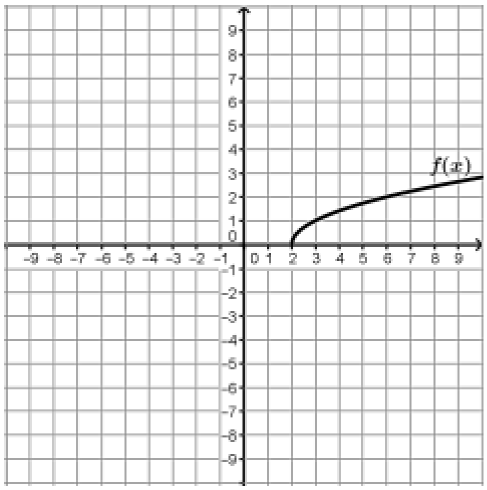
	Describe Operations on *x*	Describe Solving for *x*
1. $3x = 24$	**Multiply by 3.**	**Divide by 3 on both sides.**
2. $\frac{x}{5} = -2$		
3. $x + 17 = 20$		
4. $\sqrt{x} = 6$		
5. $\sqrt[3]{(x+1)} = 2$		
6. $x^4 = 81$		
7. $(x-9)^2 = 49$		

8. How are the descriptions for each problem above related? What similarities do you see in the operations being used on *x* and how to solve for *x*? What about the order?

Topic: Writing square root functions and finding the inverse.

Find the inverse equation and state the domain and range of the original function and its inverse.

9.



Equation for $f(x)$: $f(x) = \sqrt{x-2}$

Domain of $f(x)$:

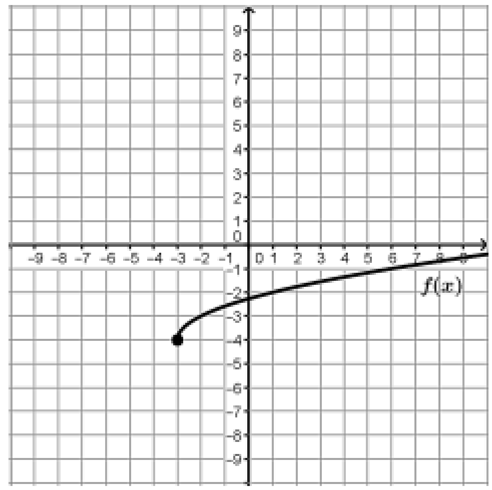
Range of $f(x)$:

Equation for $f^{-1}(x)$:

Domain of $f^{-1}(x)$:

Range of $f^{-1}(x)$:

10.



Equation for $f(x)$: $f(x) = \sqrt{x+3} - 4$

Domain of $f(x)$:

Range of $f(x)$:

Equation for $f^{-1}(x)$:

Domain of $f^{-1}(x)$:

Range of $f^{-1}(x)$:

Set

Topic: Linear functions and their inverses

Carlos and Clarita have a pet sitting business. When they were trying to decide how many dogs and how many cats they could fit into their yard, they made a table based on the following information. Cat pens require 6 ft^2 of space and the dog runs require 24 ft^2. Carlos and Clarita have up to 360 ft^2 available in the storage shed for pens and runs, while still leaving enough room to move around the cages.

They quickly realized that they could have 4 cats for each dog, so they counted the number of cats by 4.

Cats	0	4	8	12	16	20	24	28	32	36	40	44	48	52	56	60
Dogs	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1	0

11. Use the information in the table to write 5 ordered pairs that have cats as the independent variable and dogs as the dependent variable.

12. Write an explicit equation that shows how many dogs Carlos & Clarita can accommodate based on how many cats they have. (The number of dogs "d" will be a function of the number of cats "c" or $d = f(c)$.)

13. Use the information in the table to write 5 ordered pairs that have dogs as the independent variable and cats as the dependent variable.

14. Write an explicit equation that shows how many cats Carlos & Clarita can accommodate based on how many dogs they have. (The number of cats "c" will be a function of the number of dogs "d" or $c = g(d)$.)

15. Look back at questions 11 and 13. Describe how the ordered pairs are different.

16. Look back at the equations you wrote in questions 12 and 14. What relationships do you see between them? Hint: Consider the numbers in the equations.

17. What do the domain and range for each equation you wrote in questions 12 and 14 represent?

Go

Topic: Using function notation to evaluate a function.

The functions $f(x)$, $g(x)$, and $h(x)$ are defined below.

$f(x) = x + 4$ $g(x) = 5x - 12$ $h(x) = x^2 + 4x - 7$

Calculate the indicated function values. Simplify your answers.

18. $f(10)$

19. $f(a)$

20. $f(a + b)$

21. $g(-2)$

22. $h(10)$

23. $h(-2)$

Ready, Set, Go!

Ready

Topic: Solving for a variable.

Solve for *x*:

1. $17 = 5x + 2$

2. $2x^2 - 5 = 3x^2 - 12x + 31$

3. $11 = \sqrt{2x + 1}$

4. $\sqrt{2x^2 + x - 2} = 2$

5. $-4 = \sqrt[3]{5x + 1}$

6. $\sqrt[3]{352} = \sqrt[3]{7x^2 + 9}$

7. $5^x = 3125$

8. $9^x = 243$

9. $5^x = \frac{1}{125}$

10. $4^x = \frac{1}{32}$

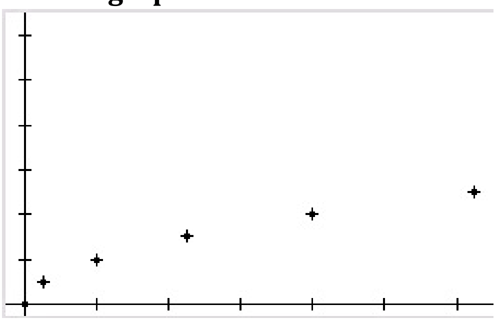
11. $3 \cdot 2^x = 96$

Set

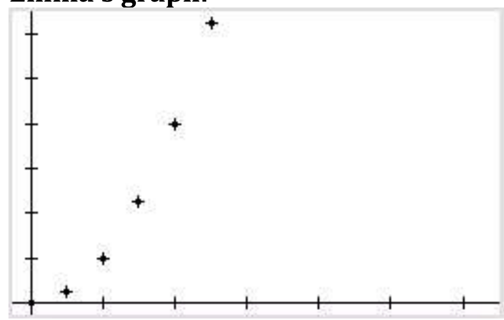
Topic: Exploring inverse functions.

12. Students were given a set of data to graph and were asked to work independently. After they had completed their graphs, each student shared his graph with his partner. When Ethan and Emma saw each other's graphs, they exclaimed together, "Your graph is wrong!" Neither graph is wrong. Explain what Ethan and Emma have done with their data.

Ethan's graph:

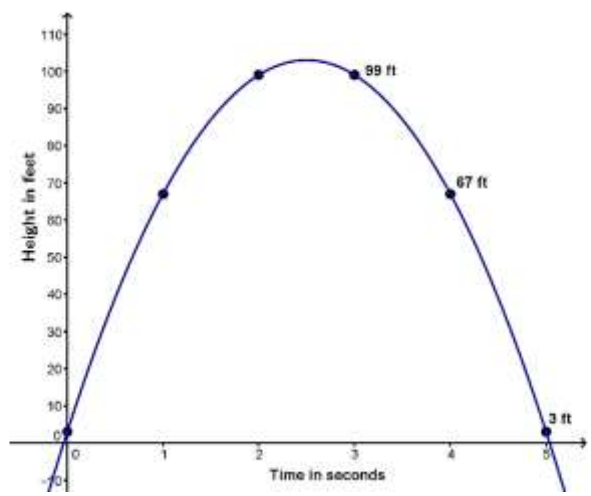


Emma's graph:



13. Describe a sequence of transformations that would take Ethan's graph onto Emma's.

14. A baseball is hit upward from a height of 3 feet and an initial velocity of 80 feet per second (about 55 mph). The graph shows the height of the ball at any second during its flight. Use the graph to answer the questions below.

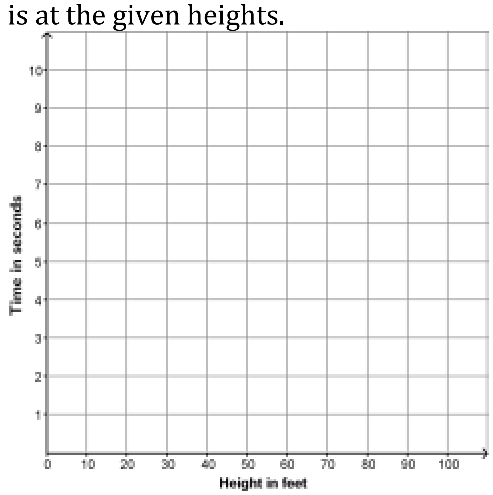


   a. Approximate the time that the ball is at its maximum height.

   b. Approximate the time that the ball hits the ground.

   c. At what time(s) is the ball 67 feet above the ground?

   d. Make a new graph that shows the time when the ball is at the given heights.



   e. Is your new graph a function? Explain.

   f. What domain restriction would make the inverse a function?

Go

Topic: Using function notation to evaluate a function.

The functions $\boldsymbol{f(x)}$, $\boldsymbol{g(x)}$, and $\boldsymbol{h(x)}$ are defined below. Use these functions for questions 15 to 23. Simplify your answers.

$\boldsymbol{f(x) = 3x}$ $\boldsymbol{g(x) = 10x + 4}$ $\boldsymbol{h(x) = x^2 - x}$

Calculate the indicated function values.

15. $f(7)$
16. $h(-9)$
17. $g(s)$
18. $g(s - t)$

Notice that the notation $\boldsymbol{f(g(x))}$ is indicating that you replace the $\boldsymbol{x}$ in $\boldsymbol{f(x)}$ with the $\boldsymbol{g(x)}$ function.
Example: $\boldsymbol{f(g(x)) = f(10x + 4) = 3(10x + 4) = 30x + 12}$

Simplify the following.

19. $f(h(x))$
20. $h(f(2))$
21. $g(f(x))$

22. Find the inverse of $f(x)$
    Reminder, the inverse of $f(x)$ is $f^{-1}(x)$.
23. Find the inverse of $g(x)$
24. If $j(x) = x^2 + 3$, find $j^{-1}(x)$
    Be sure to restrict the domain to make $j^{-1}(x)$ also a function.

Ready, Set, Go!

Ready

Topic: Solving exponential equations

Solve for the value of *x*.

1. $5^{x+1} = 5^{2x-3}$
2. $7^{3x-2} = 7^{-2x+8}$
3. $4^{3x} = 2^{2x-8}$
4. $3^{5x-5} = 9^{2x-3}$
5. $8^{x+1} = 2^{2x+3}$
6. $25^{x} = \frac{1}{125}$
7. $3^{x+1} = \frac{1}{81}$

Set

Topic: Writing the logarithmic form of an exponential equation.

Definition of Logarithm:

For all positive numbers b, where $b \neq 1$, and all positive numbers x, $\boldsymbol{y = \log_b x}$ means the same as $\boldsymbol{x = b^y}$.
*(Note the **base** of the exponent and the **base** of the logarithm are both **b**.)*

8. Why is it important that the definition of logarithms states that the base of the logarithm does not equal 1?

9. Why is it important that the definition states that the base of the logarithm is positive?

10. Why is it necessary that the definition states that $\boldsymbol{x}$ in the expression $\boldsymbol{\log_b x}$ is positive?

Write the following exponential equations in logarithmic form and the given logarithmic equation in exponential form.

Exponential form	**Logarithmic form**
11. $5^4 = 625$	
12.	$\log_3 9 = 2$
13. $\left(\frac{1}{2}\right)^{-3} = 8$	
14.	$\log_{10} 10000 = 4$
15. $4^{-2} = \frac{1}{16}$	
16.	$\log_e e^7 = 7$
17. $a^y = x$	
18. Compare the exponential form of an equation to the logarithmic form of an equation. What part of the exponential equation is the **answer** to the logarithmic equation?	

Go

Topic: Evaluating functions.

The functions $f(x)$, $g(x)$, and $h(x)$ are defined below.

$\boldsymbol{f(x) = -2x}$ $\boldsymbol{g(x) = 2x + 5}$ $\boldsymbol{h(x) = x^2 + 3x - 10}$

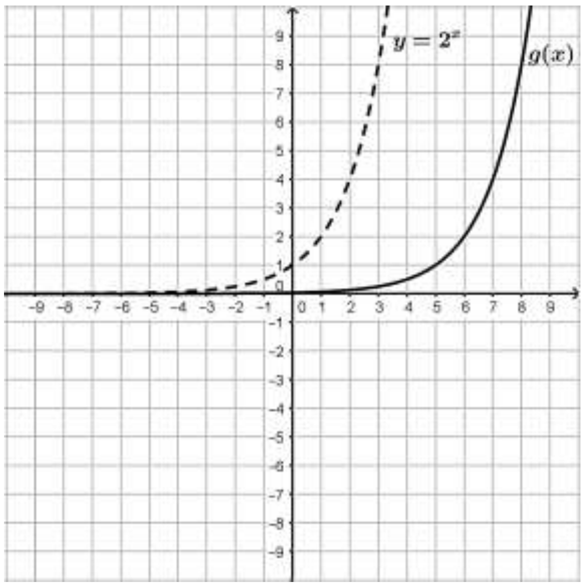
Calculate the indicated function values. Simplify your answers.

19. $f(a)$
20. $f(b^2)$
21. $g^{-1}(x)$
22. $f(g(x))$

Topic: Transformations of exponential functions

Describe the transformations of each graph from the one given. Then, write the equation of the graph.

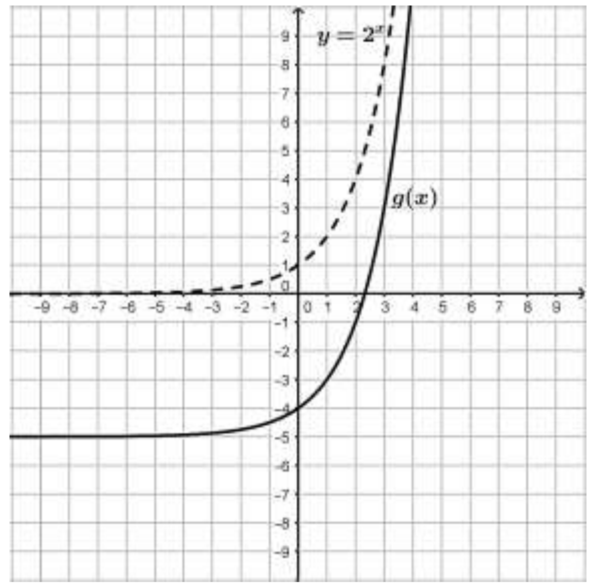
23.



Description:

$g(x) =$

24.



Description:

$g(x) =$

Name

Ready, Set, Go!

Ready

Topic: Properties of exponents

Use the product rule or the quotient rule to simplify. Leave all answers in exponential form with only positive exponents. Express solutions using the smallest base possible.

Reminder: $\boldsymbol{x^3 \cdot x^5 = (x \cdot x \cdot x) \cdot (x \cdot x \cdot x \cdot x \cdot x) = x^8}$ $\boldsymbol{x^3 \cdot x^5 = x^{(3+5)} = x^8}$	$\boldsymbol{\frac{x^3}{x^5} = \frac{\cancel{x}\cdot\cancel{x}\cdot\cancel{x}}{(\cancel{x}\cdot\cancel{x}\cdot\cancel{x}\cdot x\cdot x)} = \frac{1}{x^2}}$ $\boldsymbol{\frac{x^3}{x^5} = x^{(3-5)} = x^{-2} = \frac{1}{x^2}}$

1. $3^6 \cdot 3^5$
2. $2^3 \cdot 2^7$
3. $7^2 \cdot 7^6$
4. $10^{-4} \cdot 10^7$
5. $5^9 \cdot 5^{-6}$
6. $p^2 p^5$
7. $2^6 \cdot 2^{-3} \cdot 2$
8. $b^{11} b^{-5}$
9. $\frac{7^5}{7^2}$
10. $\frac{9^8}{9}$
11. $\frac{3^5}{3^8}$
12. $\frac{7^{-4}}{7^{-8}}$
13. $\frac{p^{-3}}{p^5}$
14. $\frac{x^7}{x^{-4}}$

Set

Topic: Inverse functions

15. Given the functions $f(x) = \sqrt{x} - 1$ and $g(x) = x^2 + 7$

   a. Calculate $f(16)$ and $g(3)$

   b. Write $f(16)$ as an ordered pair. Write $g(3)$ as an ordered pair.

   c. What do your ordered pairs for $f(16)$ and $g(3)$ imply?

   d. Find $f(25)$.

   e. Based on your answer for$f(25)$, predict $g(4)$.

   f. Find $g(4)$. Did your answer match your prediction?

   g. Are $f(x)$ and $g(x)$ inverse functions? Justify your answer.

Match the function in the left column with its inverse in the right column.

$f(x)$		$f^{-1}(x)$
__16. $f(x) = 3x + 5$	a.	$f^{-1}(x) = \log_5 x$
__17. $f(x) = x^5$	b.	$f^{-1}(x) = \sqrt[3]{x}$
__18. $f(x) = \sqrt[5]{x - 3}$	c.	$f^{-1}(x) = \frac{x-5}{3}$
__19. $f(x) = x^3$	d.	$f^{-1}(x) = \frac{x}{3} - 5$
__20. $f(x) = 5^x$	e.	$f^{-1}(x) = \log_3 x$
__21. $f(x) = 3(x + 5)$	f.	$f^{-1}(x) = x^5 + 3$
__22. $f(x) = 3^x$	g.	$f^{-1}(x) = \sqrt[5]{x}$

Go

Topic: Composite functions and inverses

Calculate $f(g(x))$ and $g(f(x))$ for each pair of functions. (Note: the notation $(f \circ g)(x)$ and $(g \circ f)(x)$ mean the same thing as $f(g(x))$ and $g(f(x))$, respectively.)

23. $f(x) = 2x + 5$ $\quad g(x) = \frac{x-5}{2}$

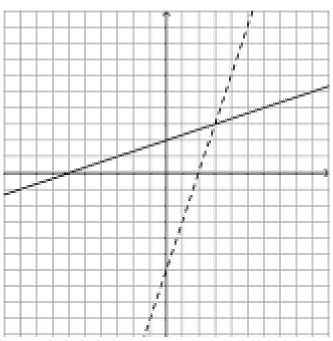
24. $f(x) = (x+2)^3$ $\quad g(x) = \sqrt[3]{x} - 2$

25. $f(x) = \frac{3}{4}x + 6$ $\quad g(x) = \frac{4(x-6)}{3}$

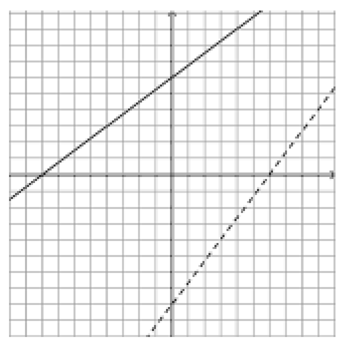
26. $f(x) = 3x - 6$ $\quad g(x) = \frac{1}{3}x + 2$

Match the pairs of functions above (#23-26) with their graphs. Label $f(x)$ and $g(x)$.

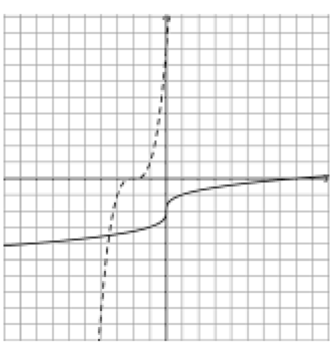
a.



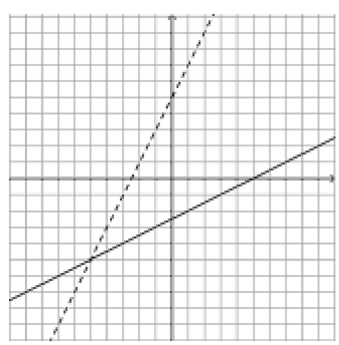
b.



c.



d.



27. Graph the line $y = x$ on each of the graphs. What do you notice?

28. Do you think your observations about the graphs in #27 has anything to do with the answers you got when you found $f(g(x))$ and $g(f(x))$? Explain.

29. Look at graph *b*. Shade the 2 triangles made by the *y*-axis, *x*-axis, and each line. What is interesting about these two triangles?

30. Shade the 2 triangles in graph *d*. Are they interesting in the same way? Explain.

31. What do you notice about your calculated values of $f(g(x))$ and $g(f(x))$ in questions 23-26? How does this relate back to question 27?

Name **Functions and their Inverses | 1.5**

Ready, Set, Go!

Ready

Topic: Properties of exponents

Use properties of exponents to simplify the following. Write your answers in exponential form with positive exponents. Use the smallest base possible in your solution.

1. $\sqrt[5]{32} \cdot \sqrt{9} \cdot \sqrt[3]{27}$

2. $\sqrt[4]{8} \cdot \sqrt[3]{16} \cdot \sqrt[6]{2}$

3. $(5^2)^3$

4. $\sqrt[7]{x^2} \cdot \sqrt[7]{x^3}$

5. $\sqrt[3]{x} \cdot \sqrt[4]{x} \cdot \sqrt[6]{x}$

6. $\sqrt[6]{a} \cdot \sqrt[3]{a^2} \cdot \sqrt[5]{b^3}$

Set

Topic: Writing inverses

Write the inverse of each function below. Be sure to restrict the domain where necessary.

7. $f(x) = 3x - 6$

8. $g(x) = \frac{2}{3}x + 12$

9. $h(x) = \sqrt{x + 7}$

Topic: Representations of inverse functions

Write the inverse of the given function in the same format as the given function.

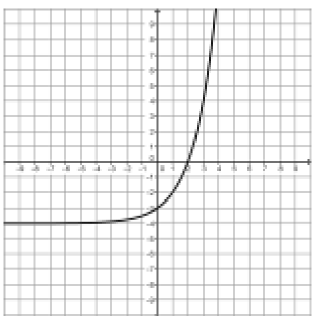
Function $f(x)$	**Inverse $f^{-1}(x)$**

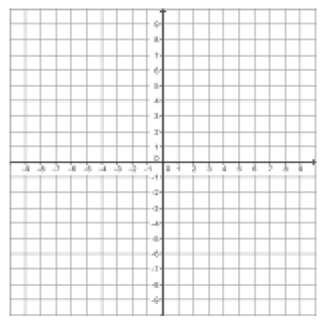
10.

x	$f(x)$
-8	0
-4	3
0	6
4	9
8	12

x	$f^{-1}(x)$

11.

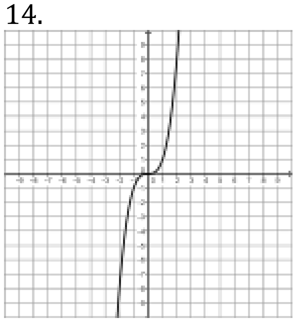


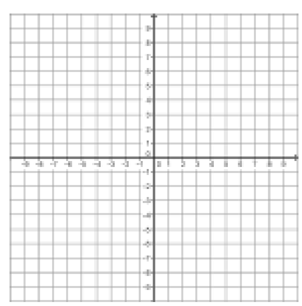


12. $f(x) = -2x + 4$

13. $f(x) = \log_3 x$

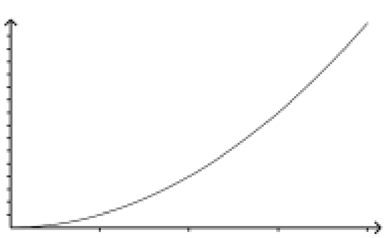
14.



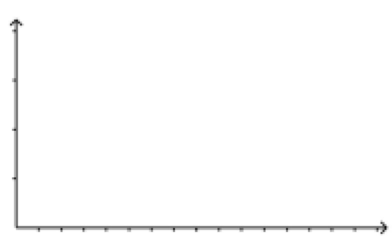


15.

x	$f(x)$
0	0
1	1
2	4
3	9
4	16



x	$f^{-1}(x)$



Go

Topic: Composite functions

Calculate $f(g(x))$ and $g(f(x))$ for each pair of functions.
(Note: the notation $(f \circ g)(x)$ and $(g \circ f)(x)$ mean the same thing, respectively.)

16. $f(x) = 3x + 7; g(x) = -4x - 11$

17. $f(x) = -4x + 60;\ \ g(x) = -\frac{1}{4}x + 15$

18. $f(x) = 10x - 5;\ \ g(x) = \frac{2}{5}x + 3$

19. $f(x) = -\frac{2}{3}x + 4;\ \ g(x) = -\frac{3}{2}x + 6$

20. Look back at your calculations for $f(g(x))$ and $g(f(x))$. Two of the pairs of equations are inverses of each other. Which ones do you think they are? Why?

21. Given $f(x) = 2^x$.

a. What is $f^{-1}(x)$?

b. What would $f\left(f^{-1}(x)\right)$ look like before it is simplified?

22. Complete the table using the definition of a logarithm: $b^x = a \iff \log_b a = x$

Exponential From	Logarithmic Form
$2^x = 7$	
	$\log_4 9 = x$
$5^{a+b} = 10$	
	$\log_{30} x = 8$
$6^{20} = (2x + 8)$	
	$\log_x 12 = 20$

Integrated Math 3
Module 2
Logarithmic Functions

Adapted from

The Mathematics Vision Project:
Scott Hendrickson, Joleigh Honey, Barbara Kuehl,
Travis Lemon, Janet Sutorius

Module 2 Overview

Prerequisite Concepts & Skills:

- Finding inverses
- Know that inverses are reflections over $y = x$
- Graph and translate exponential functions
- Transformations of functions
- Domain and range
- Solving exponential equations by making bases equal
- Simple interest

Summary of the Concepts & Skills in Module 2:

- Evaluating logarithmic expressions
- Graphing logarithmic functions
- Using proper logarithmic notation
- Expanding and condensing logarithms using logarithmic properties
- Solving logarithmic equations
- Solving exponential equations using logarithms

Content Standards and Standards of Mathematical Practice Covered:

- Content Standards: F.BF.5, F.LE.4, F.LE.4.1, F.LE.4.2, F.LE.4.3, F.IF.7e, F.IF.8
- Standards of Mathematical Practice:
  1. Make sense of problems & persevere in solving them
  2. Reason abstractly & quantitatively
  3. Construct viable arguments & critique the reasoning of others
  4. Model with mathematics
  5. Use appropriate tools strategically
  6. Attend to precision
  7. Look for & make use of structure
  8. Look for & express regularity in repeated reasoning

Module 2 Vocabulary:

- Logarithmic functions
- Base
- Argument of a logarithm
- Log of a product rule
- Log of a quotient rule
- Log of a power rule
- Compound interest
- Principal
- Growth
- Decay
- Equality property of logarithms

Concepts Used in the Next Module:

- Graphing polynomial functions
- Features of polynomial functions
- Expanding binomials

Module 2 – Logarithmic Functions

2.1 *Evaluate and compare logarithmic expressions* **(F.BF.5, F.LE.4)**
Warm Up: Inverses and Solving Exponential Equations
Classroom Task: Log Logic – A Develop Understanding Task
Ready, Set, Go Homework: Logarithmic Functions 2.1

2.2 *Graph logarithmic functions with transformations* **(F.BF.5, F.IF.7e)**
Warm Up: Graphing Exponential and Logarithmic Functions
Classroom Task: Falling Off A Log – A Solidify Understanding Task
Ready, Set, Go Homework: Logarithmic Functions 2.2

2.3 *Graph logarithmic functions with transformations* **(F.BF.5, F.IF.7e)**
Warm Up: Graphing Logarithmic Functions
Classroom Task: Falling Off Another Log – A Solidify Understanding Task
Ready, Set, Go Homework: Logarithmic Functions 2.3

2.4 *Develop understanding of the properties of logarithms* **(F.IF.8, F.LE.4, F.LE.4.1 CA, F.LE.4.3 CA)**
Warm Up: Simplifying Exponents
Classroom Task: Chopping Logs (Part 1) – A Solidify Understanding Task
Ready, Set, Go Homework: Logarithmic Functions 2.4

2.5 *Develop understanding of the properties of logarithms* **(F.IF.8, F.LE.4, F.LE.4.1 CA, F.LE.4.3 CA)**
Warm Up: Transformations of Logarithmic Functions
Classroom Task: Chopping Logs (Part 2) – A Solidify Understanding Task
Ready, Set, Go Homework: Logarithmic Functions 2.5

2.6 *Use log properties to evaluate expressions* **(F.IF.8, F.LE.4, F.LE.4.3 CA)**
Warm Up: Begin Task
Classroom Task: Log-Arithm-etic – A Practice Understanding Task
Ready, Set, Go Homework: Logarithmic Functions 2.6

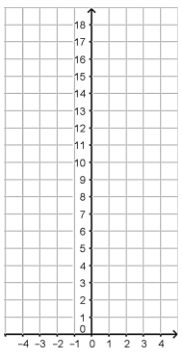
27 *Solve exponential and logarithmic equations in base 10 using technology* **(F.LE.4, F.LE.4.2 CA, F.LE.4.3 CA)**
Warm Up: Change of Base
Classroom Task: Powerful Tens – A Practice Understanding Task
Ready, Set, Go Homework: Logarithmic Functions 2.7

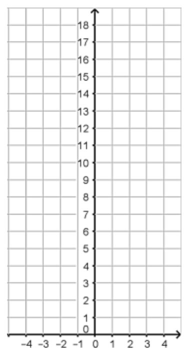
2.8 *Module 2 Review* **(F.LE.4, F.LE.4.3 CA, F.BF.5, F.IF.7e, F.IF,8)**
Warm Up: Exponential Equations
Classroom Task: Log Rolling
Ready, Set Homework: Logarithmic Functions 2.8

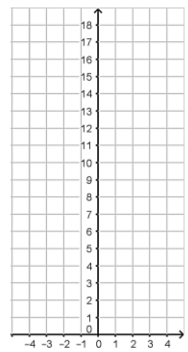
2.1 Warm Up
Inverses and Exponential Equations

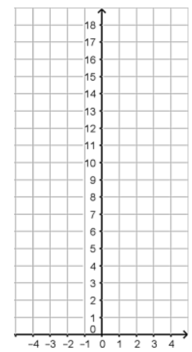
Graph each function over the domain $\{-4 \leq x \leq 4\}$.

1. $y = 3^x$
2. $y = 2 \cdot 3^x$
3. $y = \left(\frac{1}{3}\right)^x$
4. $y = 4\left(\frac{1}{3}\right)^x$









For questions 5-6, solve each equation.

5. $2^x = 64$

6. $7^{3x+2} = 2401$

2.1 Log Logic
A Develop Understanding Task

We began thinking about logarithms as inverse functions for exponentials in *Tracking the Tortoise*. Logarithmic functions are interesting and useful on their own. In the next few tasks, we will be working on understanding logarithmic expressions, logarithmic functions, and logarithmic operations on equations.

We showed the inverse relationship between exponential and logarithmic functions using a diagram like the one below:

Input $\qquad\qquad\qquad\qquad\qquad\qquad$ Output

$$x = 3 \xrightarrow{f(x) = 2^x} 2^3 = 8 \xrightarrow{f^{-1}(x) = \log_2 x} 3$$

We could summarize this relationship by saying:

$2^3 = 8$ so, $\log_2 8 = 3$

Logarithms can be defined for any base used for an exponential function. Base 10 is common. Using base 10, you can write statements like these:

$10^1 = 10$ so, $\log_{10} 10 = 1$

$10^2 = 100$ so, $\log_{10} 100 = 2$

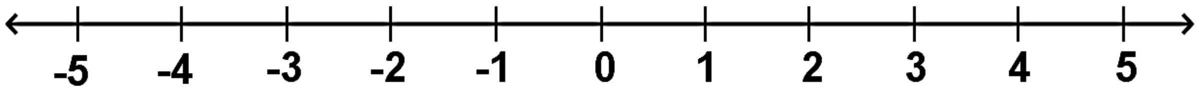
$10^3 = 1000$ so, $\log_{10} 1000 = 3$

The notation is a little strange, but you can see the inverse pattern of switching the inputs and outputs.

The next few problems will give you an opportunity to practice thinking about this pattern and possibly make a few conjectures about other patterns that you may notice with logarithms.

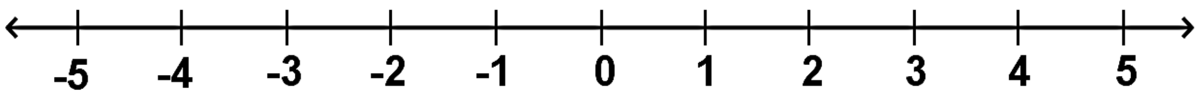
Place/label the following expressions on the number line. Use the space below the number line to explain how you knew where to place each expression.

1. A. $\log_3 3$ B. $\log_3 9$ C. $\log_3 \frac{1}{3}$ D. $\log_3 1$ E. $\log_3 \frac{1}{9}$



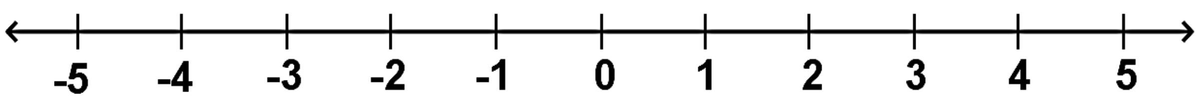
Explain: ____

2. A. $\log_3 81$ B. $\log_{10} 100$ C. $\log_8 8$ D. $\log_5 25$ E. $\log_2 32$



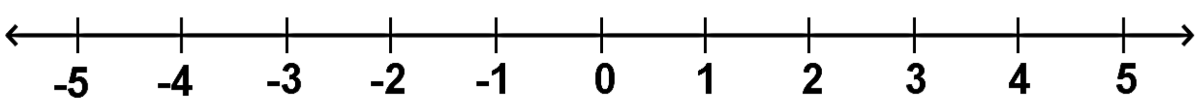
Explain: ____

3. A. $\log_7 7$ B. $\log_9 9$ C. $\log_{11} 1$ D. $\log_{10} 1$



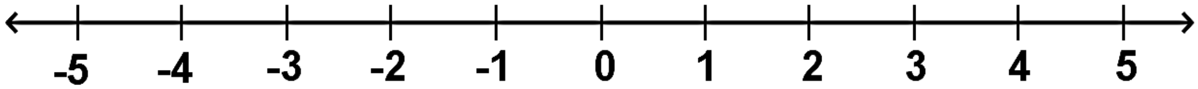
Explain: ____

4. A. $\log_2 \left(\frac{1}{4}\right)$ B. $\log_{10} \left(\frac{1}{1000}\right)$ C. $\log_5 \frac{1}{25}$ D. $\log_6 \frac{1}{6}$



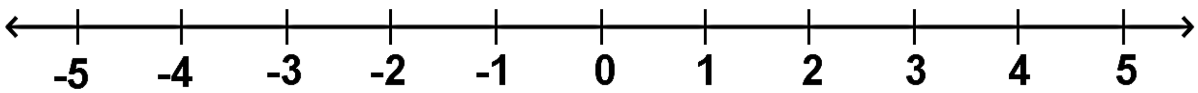
Explain: ____

5. A. $\log_4 16$ B. $\log_2 16$ C. $\log_8 16$ D. $\log_{16} 16$



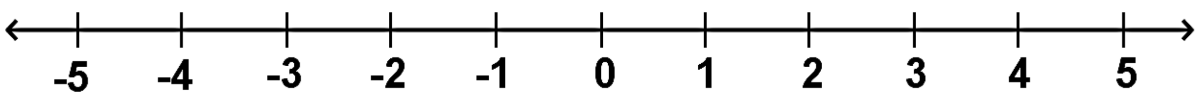
Explain: ____

6. A. $\log_2 5$ B. $\log_5 10$ C. $\log_6 1$ D. $\log_5 5$ E. $\log_{10} 5$



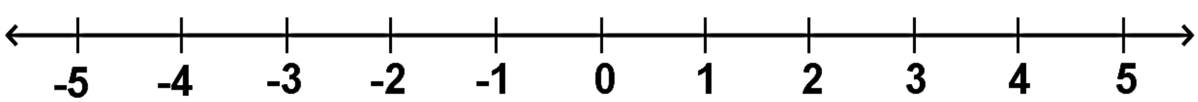
Explain: ____

7. A. $\log_{10} 50$ B. $\log_{10} 150$ C. $\log_{10} 1000$ D. $\log_{10} 500$



Explain: ____

8. A. $\log_{\frac{1}{3}}\left(\frac{1}{3}\right)^2$ B. $\log_{\frac{2}{7}}\left(\frac{2}{7}\right)^{-2}$ C. $\log_{\frac{4}{5}}\left(\frac{4}{5}\right)^0$ D. $\log_{\frac{1}{9}}\left(\frac{1}{9}\right)^{-1}$ E. $\log_{\frac{3}{4}}\left(\frac{3}{4}\right)^3$



Explain: ____

Based on your work with logarithmic expressions, determine whether each of these statements is always true, sometimes true, or never true. If the statement is sometimes true, describe the conditions that make it true. Explain your answers.

9. The value of $\log_b x$ is positive.
   Select one: always true sometimes true never true

   Explain: ____

10. $\log_b x$ is not a valid expression if x is a negative number.
    Select one: always true sometimes true never true

    Explain: ____

11. $\log_b 1 = 0$ for
    a. $0 < b < 1$
       Select one:
       always true
       sometimes true
       never true

       Explain: ____________________

    b. $b > 1$
       Select one:
       always true
       sometimes true
       never true

       Explain: ____________________

12. $\log_b b = 1$ for
    a. $0 < b < 1$
       Select one:
       always true
       sometimes true
       never true

       Explain: ____________________

    b. $b > 1$
       Select one:
       always true
       sometimes true
       never true

       Explain: ____________________

13. $\log_2 x > \log_3 x$ for any value of x.
    Select one: always true sometimes true never true

    Explain: ____

14. $\log_b b^n = n$ for

a. $0 < b < 1$

Select one:

always true

sometimes true

never true

Explain:____

b. $b > 1$

Select one:

always true

sometimes true

never true

Explain: ____

15. What type of numbers are always allowed to be a base? Explain.

16. What type of numbers are always allowed to be an argument? Explain.

2.2 Warm Up
Graphing Exponential and Logarithmic Functions

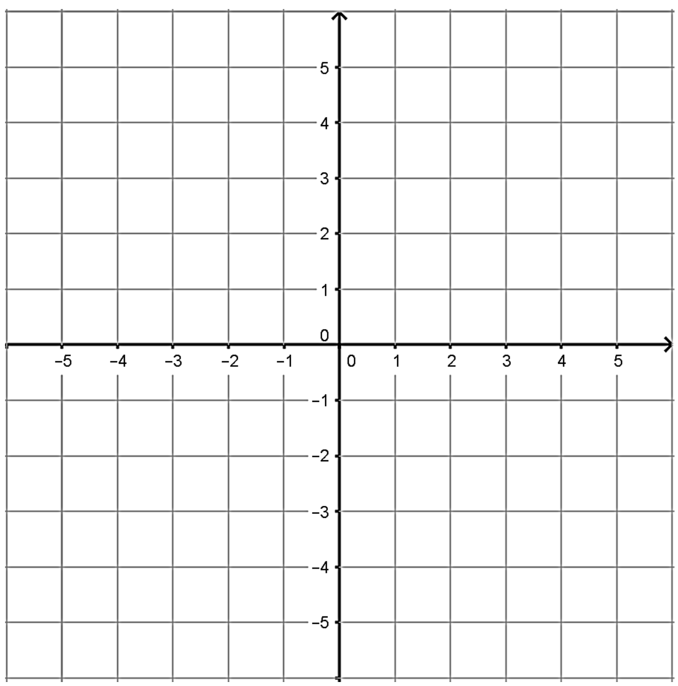
Complete the table of values to graph the exponential function. Then fill in the table of values for its inverse function (logarithmic function). Graph and label both functions on the same set of axes.

$f(x) = 2^x$

x	$f(x)$
-2	
-1	
0	
1	
2	

$f^{-1}(x) = \log_2 x$

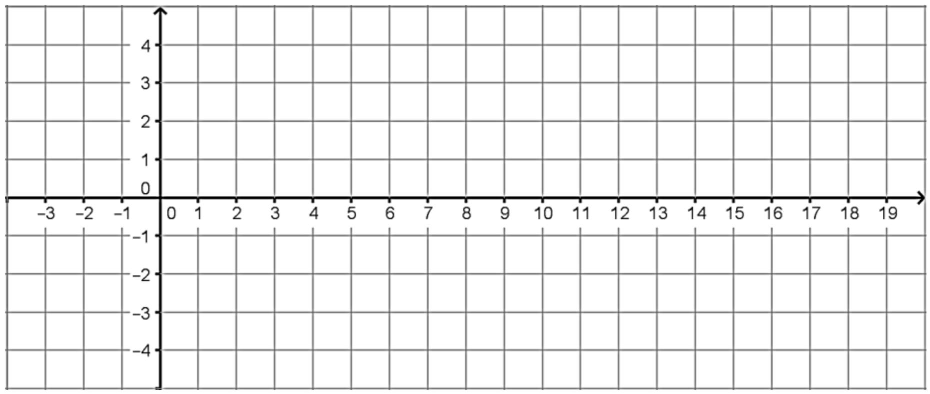
x	$f(x)$



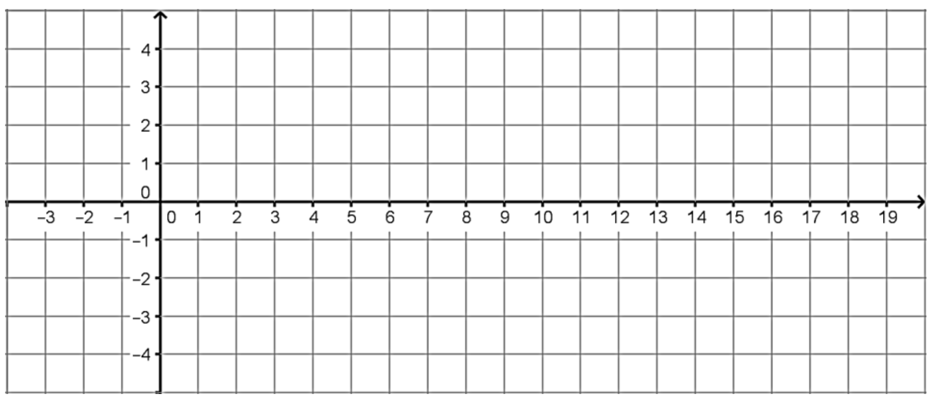
2.2 Falling Off A Log
A Solidify Understanding Task

1. Construct a table of values and a graph for each of the following functions. Be sure to select at least two values in the interval $0 < x < 1$.

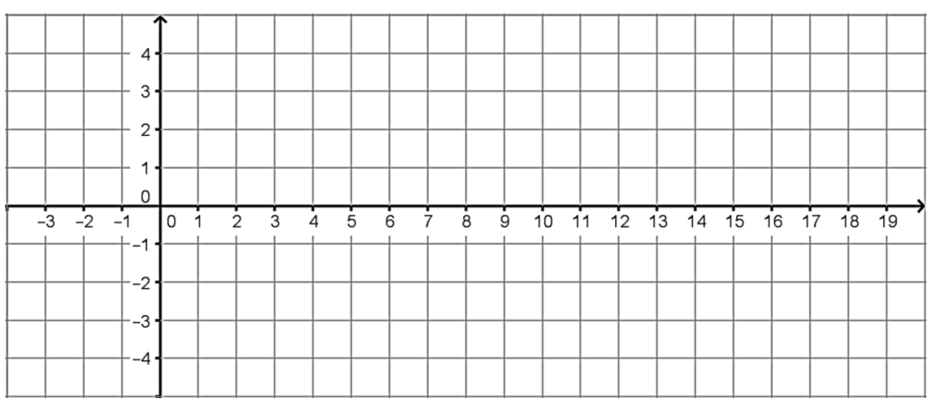
a. $f(x) = \log_3 x$



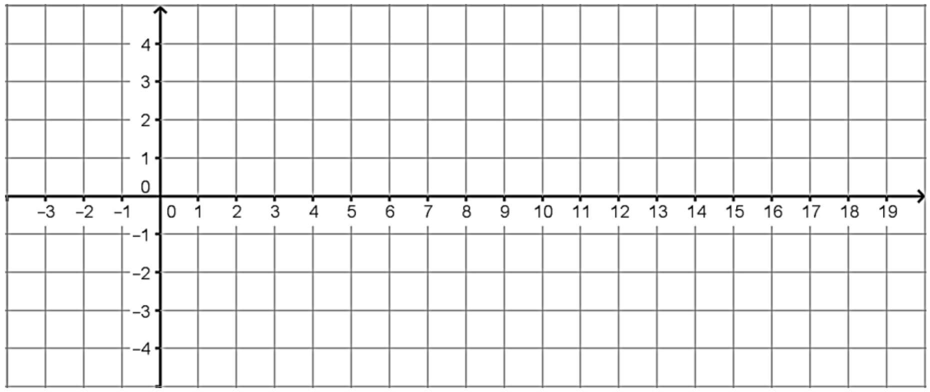
b. $g(x) = \log_4 x$



c. $p(x) = \log_{10} x$



d. $q(x) = \log_{\frac{1}{2}} x$



2. How did you decide what values to use for *x* in your table?

3. How did you use the *x*-values to find the *y*-values in the table?

4. What similarities do you see in the graphs?

5. What differences do you observe in the graphs?

6. How does changing the base on a logarithm affect the graph?

Let's focus now on $p(x) = \log_{10} x$ so that we can use technology to observe the effects of changing parameters on the function. Because base 10 is a very commonly used base for exponential and logarithmic functions, it is called the **common logarithm** and written without the base, like this: $p(x) = \log x$.

7. Use technology to graph $y = \log x$. How does the graph compare to the graph that you constructed in question 1c?

8. How do you predict that the graph of $y = \log x + k$ will be different from the graph of $y = \log x$?

9. Test your prediction by graphing $y = \log x + k$ for various values of k. What is the effect of k? Make a general argument for why this would be true for all logarithmic functions.

10. How do you predict that the graph of $y = \log(x - h)$ will be different from the graph of $y = \log x$?

11. Test your prediction by graphing $y = \log(x - h)$ for various values of h.
    a. What is the effect if h is positive?

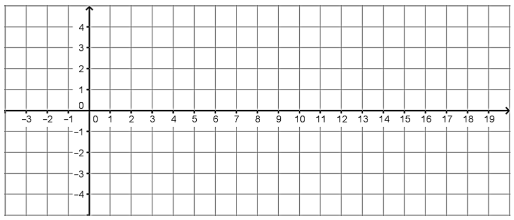
    b. What will be the effect if h is negative?

    c. Make a general argument for why this is true for all logarithmic functions.

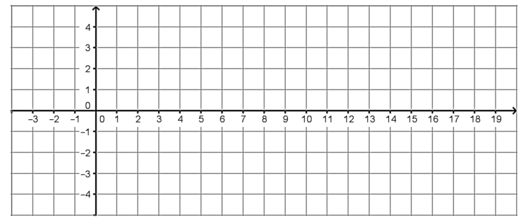
2.3 Warm Up
Graphing Logarithmic Functions

Graph each logarithmic function below.

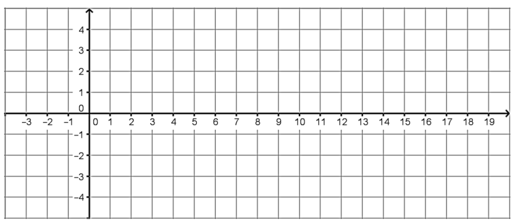
1. $f(x) = \log_2(x)$



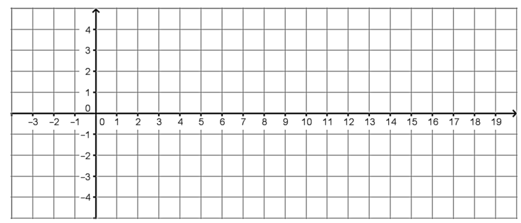
2. $g(x) = 1 + \log_2(x)$



3. $f(x) = \log_2(x - 1)$



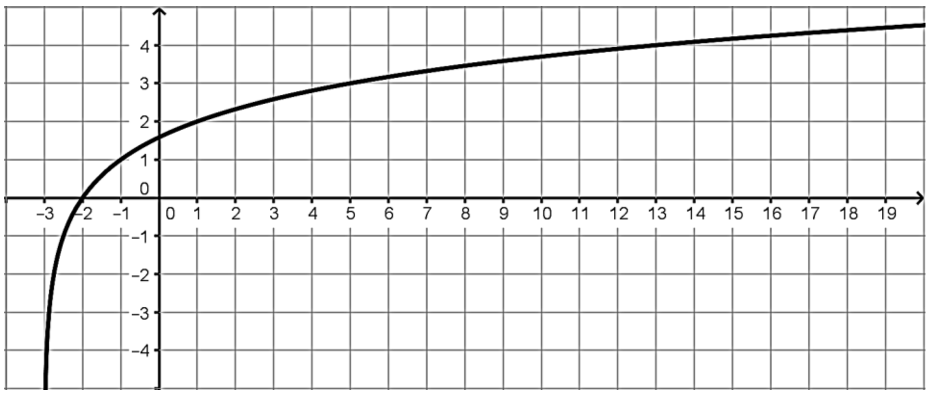
4. $g(x) = 3 + \log_2(x + 2)$



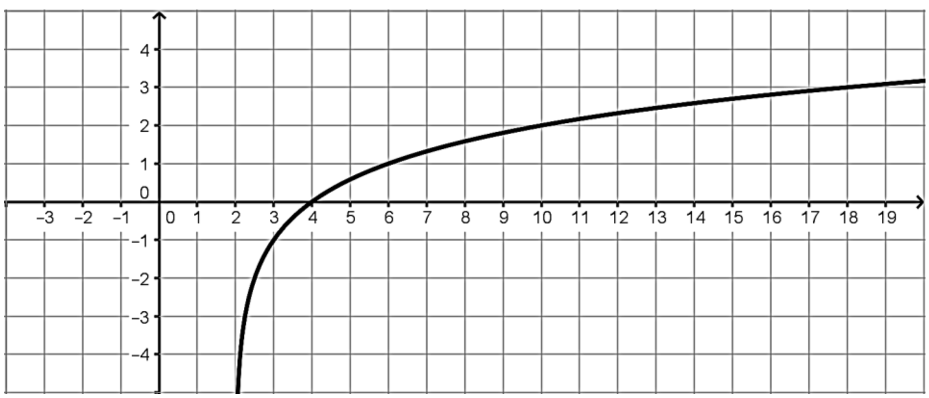
2.3 Falling Off Another Log
A Solidify Understanding Task

1. Write an equation for each of the following functions. These graphs are transformations of $f(x) = \log_2 x$, which you graphed in the warm up.

   a.

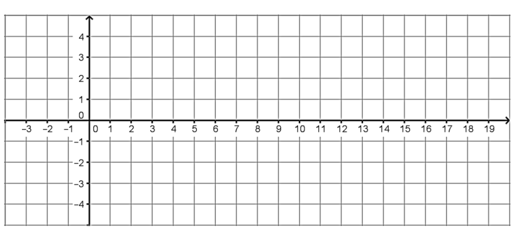


   b.

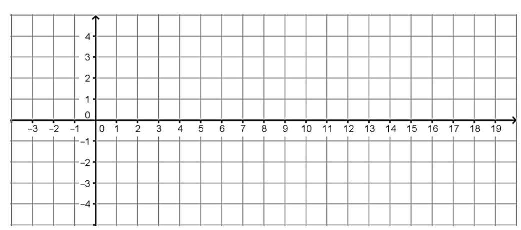


2. Graph and label each of the following functions:

   a. $f(x) = \log_2(x + 1) - 3$



   b. $g(x) = 1 + \log_2(x - 4)$



3. How do the transformations of logarithmic functions change the domain, range, and asymptotes?

4. Without graphing, describe the vertical and horizontal translations of $f(x) = \log_3(x)$.
   a. $f(x) = \log_3(x - 4) + 2$
   b. $g(x) = 5 - \log_3(x + 7)$

5. Carlos wants to graph $y = \log_3(x)$. He starts by creating the table below, but is stuck because he does not have a calculator.

   a. What advice would you give to him about his chosen x-values in the table?

x	-2	-1	0	1	2	3	4	5	6	7
y										

   b. Create a table of values for $y = \log_3(x)$ that can be completed without the use of a calculator.

x							
y							

2.4 Warm Up
Simplifying Exponents

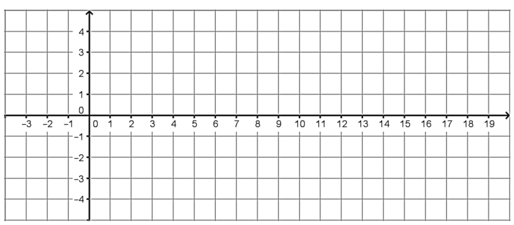
Simplify each expression using properties of exponents.

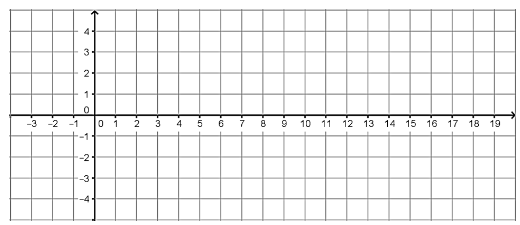
1. $(2x^5y^3)(8x^2y^7)$

2. $(3x^6y^4)^2$

3. $\frac{18x^6y^4}{24x^8y}$

4. Graph and label each of the following functions:
   a. $f(x) = \log_5(x+2) - 2$
   b. $g(x) = 3 + \log_4(x-4)$





2.4 Chopping Logs (Part 1)
A Solidify Understanding Task

Abe and Mary are working on their math homework together when Abe has a brilliant idea

Abe: I was just looking at this log function that we graphed in *Falling Off A Log*:
$y = \log_2(x + b)$

I started to think that maybe I could just "distribute" the log so that I get:
$y = \log_2 x + \log_2 b$

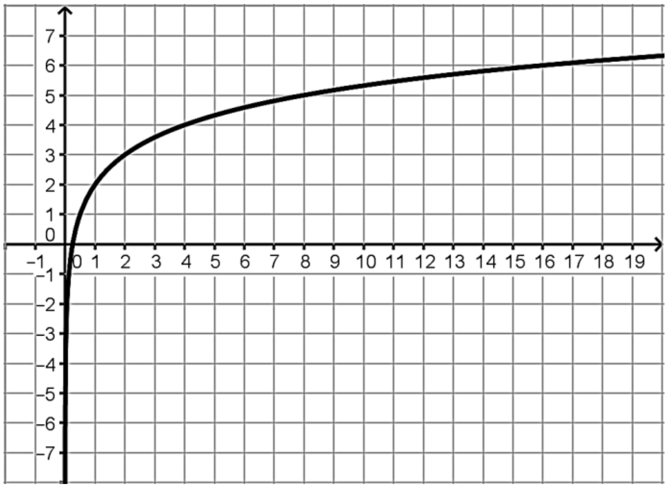
I guess I'm saying that I think these are equivalent expressions, so I could write it this way:
$\log_2(x + b) = \log_2 x + \log_2 b$

Mary: I don't know about that. Logs are tricky and I don't think that you're really doing the same thing here as when you distribute a number.

1. What do you think? How can you verify if Abe's idea works?

2. If Abe's idea works, give some examples that illustrate why it works. If Abe's idea doesn't work, give a counter-example.

Abe: I just know that there is something going on with these logs. I just graphed $f(x) = \log_2(4x)$. Here it is:



It's weird because I think that this graph is just a translation of $y = \log_2 x$. Is it possible that the equation of this graph could be written more than one way?

3. How would you answer Abe's question? Are there conditions that could allow the same graph to have different equations?

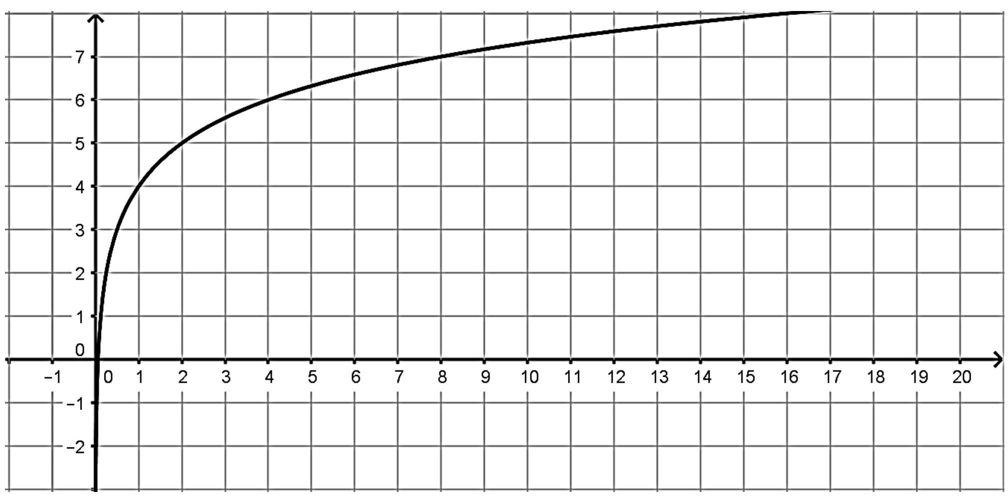
Mary: When you say, "a translation of $y = \log_2 x$" do you mean that it is just a vertical or horizontal shift? What could that equation be?

4. Find an equation for $f(x)$ that shows it to be a horizontal or vertical shift of $y = \log_2 x$.

Mary: I wonder why the vertical shift turned out to be up 2 when the *x* was multiplied by 4. I wonder if it has something to do with the power that the base is raised to, since this is a log function. Let's try to see what happens with $y = \log_2(8x)$ and $y = \log_2(16x)$.

5. Try to write an equivalent equation for each of these graphs that is a vertical shift of $y = \log_2 x$.

a. $y = \log_2(8x)$ Equivalent equation: ______________________



b. $y = \log_2(16x)$ Equivalent equation: ______________________

Mary: Oh my gosh, I think I know what is happening here. Here's what we see from the graphs:

$$\log_2(4x) = 2 + \log_2 x$$
$$\log_2(8x) = 3 + \log_2 x$$
$$\log_2(16x) = 4 + \log_2 x$$

Here's the brilliant part: We know that $\log_2 4 = 2$, $\log_2 8 = 3$, $\log_2 16 = 4$. So:

$$\log_2(4x) = \log_2 4 + \log_2 x$$
$$\log_2(8x) = \log_2 8 + \log_2 x$$
$$\log_2(16x) = \log_2 16 + \log_2 x$$

I think it looks like the "distributive" thing that you were trying to do, but since you can't really distribute a function, it's really just a log **multiplication rule**. I guess my rule would be:

$$\boldsymbol{\log_2 ab = \log_2 a + \log_2 b}$$

6. How can you express Mary's rule in words?

7. Is this statement true? If it is, give some examples that illustrate why it works. If it is not true provide a counter example.

8. Use the above property to rewrite the following expressions as a single logarithm.
   a. $3 + \log_3 x$
   b. $5 + \log_3 x$

Practice:

9. Rewrite each of the following as the sum and/or difference of logarithms (this is called expanding).
   a. $\log(100xy)$
   b. $\log_2(64x)$
   c. $\log_3(2xyz)$

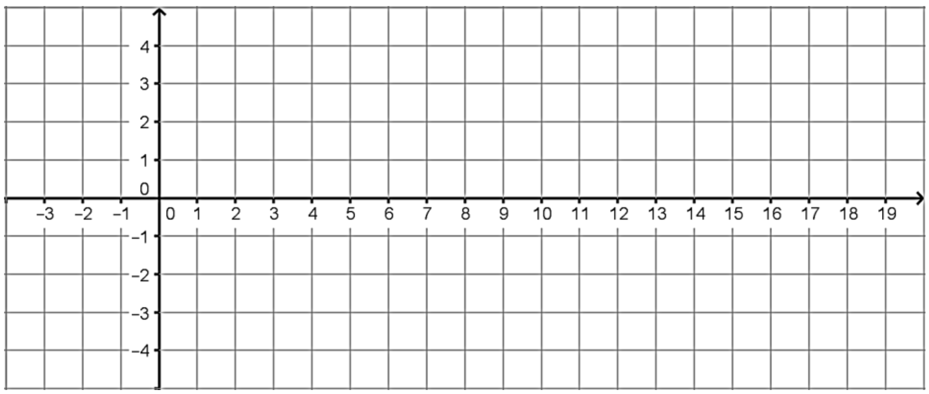
10. Rewrite each as a single logarithm (this is called condensing).
    a. $\log_5 x + \log_5 y + \log_5 z$
    b. $8 + \log_2 x + \log_2 y + \log_2 z$

2.5 Warm Up
Transformations of Logarithmic Functions

Describe the transformation of the parent function ($f(x) = \log_3 x$) for each problem. Then graph each logarithmic function and identify the domain of the function.

1. $f(x) = \log_3(x - 4)$

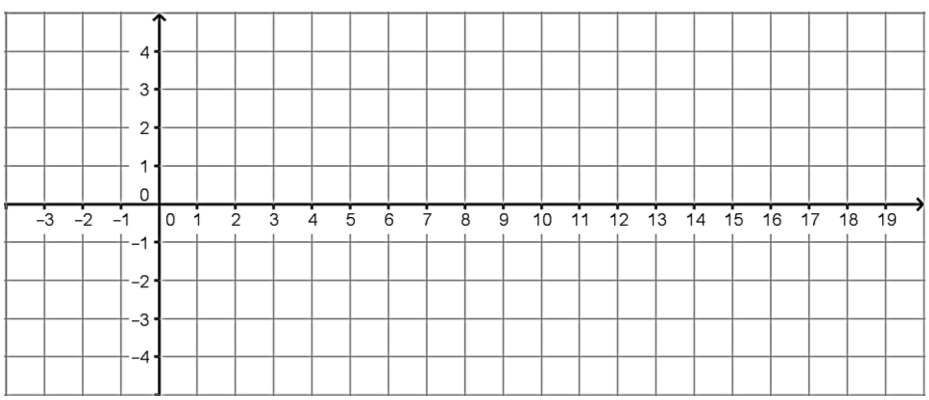
   Describe the Transformation of the Parent Function:



   Domain:

2. $f(x) = 2 + \log_3 x$

   Describe the Transformation of the Parent Function:



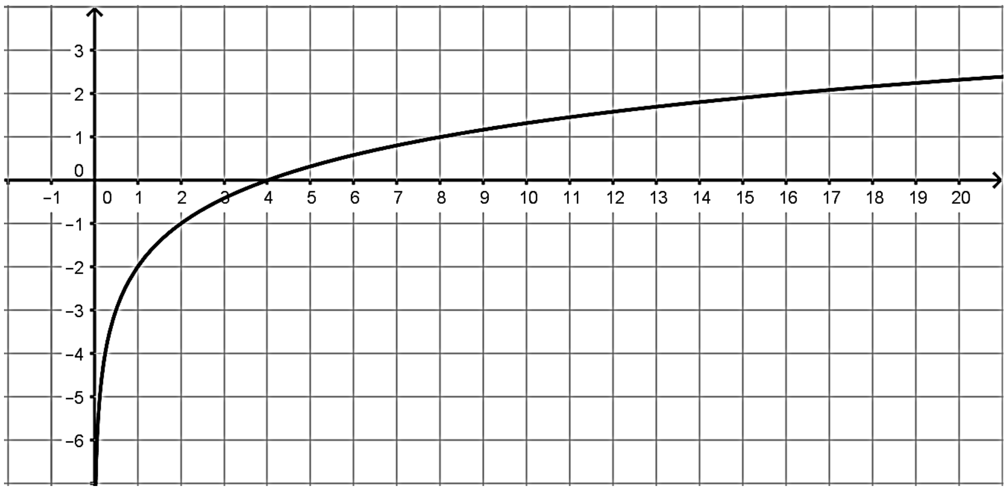
   Domain:

2.5 Chopping Logs (Part 2)
A Solidify Understanding Task

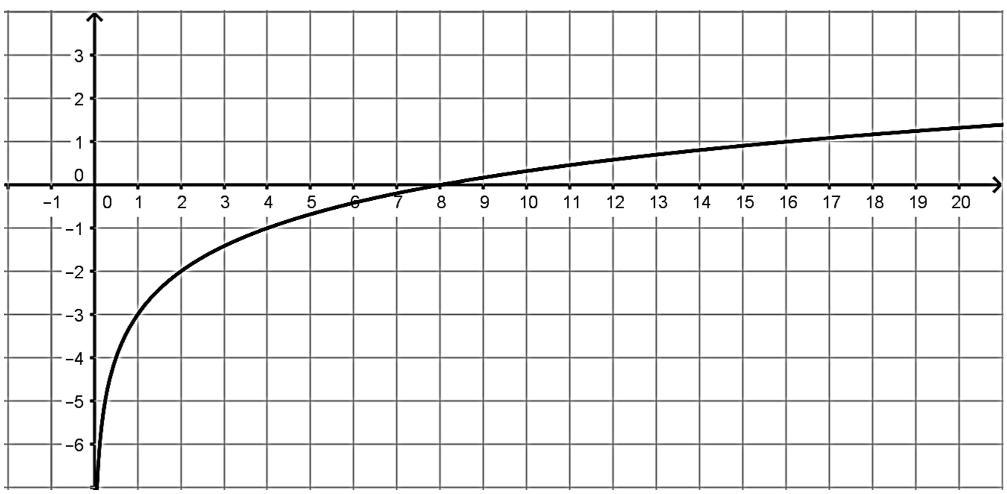
Mary: So, I wonder if a similar thing happens if you have division inside the argument of a log function. I'm going to try some examples. If my theory works, then all of these graphs will just be vertical shifts of $y = \log_2 x$.

1. Here are Abe's examples and their graphs. Test Abe's theory by trying to write an equivalent equation for each of these graphs that is a vertical shift of $y = \log_2 x$.

a. $y = \log_2 \frac{x}{4}$ Equivalent equation: ______________________



b. $y = \log_2 \frac{x}{8}$ Equivalent equation: ______________________



2. Use these examples to write a **rule for division** inside the argument of a log that is like the rule that Mary wrote for multiplication inside a log (see Mary's argument prior to question 6 in the previous task).

3. Is this statement true? If it is, give some examples that illustrate why it works. If it is not true, provide a counter-example.

4. Use what you have learned so far to fill in the blanks below that describe Abe's thought process in developing an additional log property.

Abe: You're definitely brilliant for thinking of that multiplication rule. But I'm a genius because I've used your multiplication rule to come up with a **power rule**.

Let's say that you start with:	$\log_2(x^3)$
Really that's the same as having:	$\log_2(\_\_\_\cdot\_\_\_\cdot\_\_\_)$
So, I could use your multiplying rule and write:	$\_\_\_\_ + \_\_\_\_ + \_\_\_\_$
I notice that there are 3 terms that are all the same. That makes it:	$3\cdot\_\_\_\_$
So my rule is:	$\log_2(x^3) = 3\log_2 x$

If your rule is true, then I have proven my power rule.

Mary: I don't think it's really a power rule unless it works for any power. You only showed how it might work for a power of 3.

Abe: Oh good grief! Ok, I'm going to say that it can be any number *x*, raised to any power, *k*. My power rule is:

$$\log_2(x^k) = k\cdot\log_2 x$$

5. Make an argument about Abe's power rule. Is it true or not?

Abe: Before we win the Field's Medal for mathematics, I suppose that we need to think about whether or not these rules work for any base.

6. The three rules, written for any positive base, $b \neq 1$, are:
   Log of a Product Rule: $\boldsymbol{\log_b(xy) = \log_b x + \log_b y}$

   Log of a Quotient Rule: $\boldsymbol{\log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y}$

   Log of a Power Rule: $\boldsymbol{\log_b\left(x^k\right) = k \cdot \log_b x}$

   Make an argument for why these rules will work in any positive base, $b \neq 1$, if they work for base 2.

7. How are these rules similar to the rules for exponents? Why might exponents and logs have similar rules?

Expand each logarithm as much as possible.

8. $\log_2 \frac{8x^4y^7}{z^9}$

9. $\log_6 \frac{k^5}{36m^3b^8}$

10. $\log_3 \frac{r^2}{p^7g^4}$

11. $\log_4 \frac{64k^3z^6}{27n^9}$

Condense each logarithmic expression into a single logarithm.

12. $\log_2 h + 4\log_2 p - 3\log_2 b$

13. $5\log_3 y + 2\log_3 v + \log_3 d$

14. $6\log_5 g - 2\log_5 r - 4\log_5 w$

15. $2\log_6 c - \log_6 q - 4\log_6 t + 9\log_6 x$

2.6 Log-Arithm-etic
A Practice Understanding Task

Abe and Mary are feeling good about their log rules and bragging about mathematical prowess to all their friends when this exchange occurs:

Stephen: I guess you think you're pretty smart because you figured out some log rules, but I want to know what they're good for.

Abe: Well, we've seen a lot of times when equivalent expressions are handy. Sometimes when you write an expression with a variable in it in a different way it means something different.

1. What are some examples from your previous experience where equivalent expressions were useful?

Mary: I was thinking about the *Log Logic* task where we were trying to estimate and order certain log values. I was wondering if we could use our log rules to take values we know and use them to find values that we don't know.

For instance: Let's say you want to calculate $\log_2 6$. If you know the values of $\log_2 2$ and $\log_2 3$, then you can use the product rule and say:

$$\log_2(2 \cdot 3) = \log_2 2 + \log_2 3$$

Stephen: That's great. Everyone knows that $\log_2 2 = 1$, but what is $\log_2 3$?

Abe: Oh, I saw this somewhere. Uh, $\log_2 3 = 1.585$. So Mary's idea really works. You say:

$$\begin{aligned}\log_2(2 \cdot 3) &= \log_2 2 + \log_2 3\\ &= 1 + 1.585\\ &= 2.585\\ \log_2 6 &= 2.585\end{aligned}$$

2. Based on what you know about logarithms, explain why 2.585 is a reasonable value for $\log_2 6$.

Stephen: Oh, oh! I've got one. I can figure out $\log_2 5$ like this:

$$\begin{aligned} \log_2(2+3) &= \log_2 2 + \log_2 3 \\ &= 1 + 1.585 \\ &= 2.585 \\ \log_2 5 &= 2.585 \end{aligned}$$

3. Can Stephen and Mary both be correct? Explain who's right and who's wrong (if anyone) and why.

Now you can try applying the log rules yourself. Use the values that are given and the ones that you know by definition like $\log_2 2 = 1$ to figure out each of the following values. Explain what you did in the space below each question.

$\log_2 3 = 1.585$ $\quad\quad$ $\log_2 5 = 2.322$ $\quad\quad$ $\log_2 7 = 2.807$

The three rules, written for any base $b > 1$ are:

Log of a Product Rule: $\boldsymbol{\log_b(xy) = \log_b x + \log_b y}$

Log of a Quotient Rule: $\boldsymbol{\log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y}$

Log of a Power Rule: $\boldsymbol{\log_b\left(x^k\right) = k \cdot \log_b x}$

4. $\log_2 9 =$

5. $\log_2 10 =$

6. $\log_2 12 =$

7. $\log_2\left(\frac{7}{3}\right) =$

8. $\log_2\left(\frac{30}{7}\right) =$

9. $\log_2 486 =$

10. Given the work that you have just done, what other values would you need to figure out the value of the base 2 log for any number?

Solve the following equations. Be sure to check your solutions.

11. $\log_2(x+2) = \log_2(x-5)$

12. $\log_3(x+4) = 1$

13. $\log_2 x + \log_2(x+6) = 4$

14. $\log_4(4x+7) = \log_4(11x)$

15. $2 \cdot \log x = \log 25$

16. $\log(x-4) + \log(x+1) = \log(x-8)$

Extension Problems:

Sometimes thinking about equivalent expressions with logarithms can get tricky. Consider each of the following expressions and decide if they are always true for the numbers in the domain of the logarithmic function, sometimes true, or never true. Explain your answers. If you answer "sometimes true," describe the conditions that must be in place to make the statement true.

17. $\log_4 5 - \log_4 x = \log_4 \left(\frac{5}{x}\right)$

18. $\log 3 - \log x - \log x = \log \left(\frac{3}{x^2}\right)$

19. $\log x - \log 5 = \frac{\log x}{\log 5}$

20. $5 \log x = \log x^5$

21. $2 \log x + \log 5 = \log(x^2 + 5)$

22. $\frac{1}{2} \log x = \log \sqrt{x}$

23. $\log(x - 5) = \frac{\log x}{\log 5}$

2.7 Warm Up
Change of Base

Tia and Tehani were working on their homework assignment together and were becoming frustrated with not being able to find the value of $\log_2 3$ without guess and check. Tehani said, “There has to be a way to do this!” Tia agreed, “We’ve learned how a calculator can do so many things with common logs with base 10, shouldn’t there be a way to change the equation so we can use the calculator?

Fortunately, there is a way to use properties of logarithms to find the value of a logarithm of any base. The problems below will help you understand the Change of Base formula for logarithms.

1. Find $\log_8 16 = x$ by first rewriting in exponential form.

2. In each blank below, provide an explanation for the steps in the solution to $\log_8 16 = x$.

$\log_8 16 = x$	“Original problem”
$8^x = 16$	______________________
$\log(8^x) = \log(16)$	“Taking the log of both sides”
$x \cdot \log(8) = \log(16)$	______________________
$x = \dfrac{\log(16)}{\log(8)}$	______________________
$x = \dfrac{1.204119983}{0.903089987} = \dfrac{4}{3}$	______________________

3. The previous problems both showed how to find the solution to $\log_8 16 = x$. Use the process shown in problem 2 to find the solution to $\log_2 3 = x$.

2.7 Powerful Tens
A Practice Understanding Task

Table Puzzles

1. Use the tables to find the missing values of *x*. For any value you are unable to find, write an equation that could be used to find *x*.

a.

x	$y = 10^x$
-2	$\frac{1}{100}$
1	10
	50
	100
3	1000

b.

x	$y = 3(10)^x$
	0.3
0	3
	94.87
2	300
	1503.56

c.

x	$y = \log x$
0.01	-2
	-1
10	1
	1.6
100	2

d.

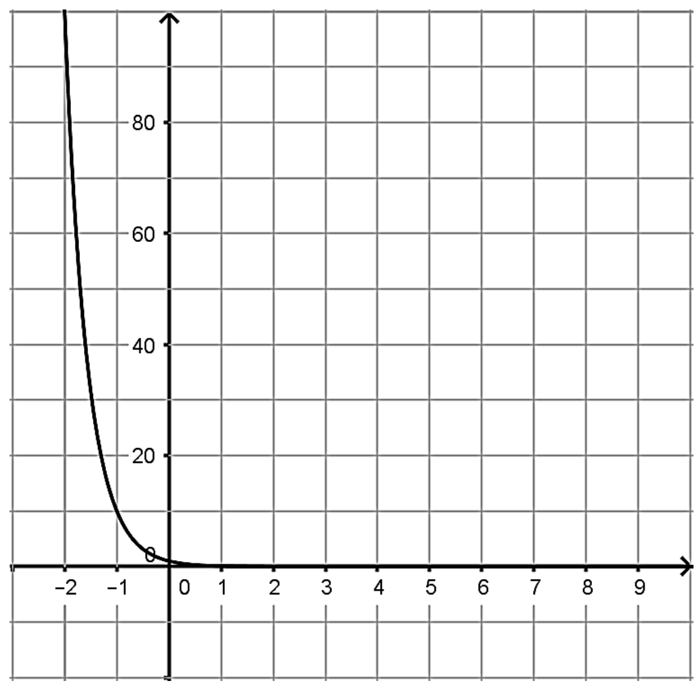
x	$y = \log(x+3)$
	-2
-2.9	-1
	0.3
7	1
	3

2. What strategy did you use to find the solutions to equations generated by the tables that contained exponential functions?

3. What strategy did you use to find the solutions to equations generated by the tables that contained logarithmic functions?

Graph Puzzles

4. The graph of $y = 10^{-x}$ is given below. Use the graph to solve the equations for x and label the solutions.



a. $40 = 10^{-x}$

x = ____________

Label the solution with an A on the graph.

b. $10^{-x} = 10$

x = ____________

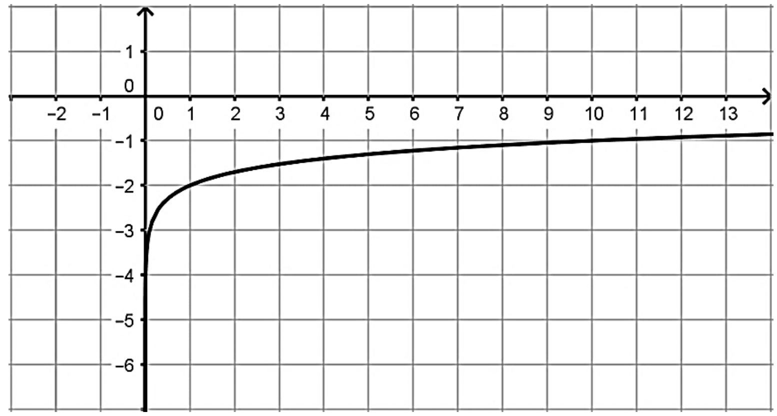
Label the solution with a B on the graph.

c. $10^{-x} = 0.1$

x = ____________

Label the solution with a C on the graph.

5. The graph of $y = -2 + \log x$ is given below. Use the graph to solve the equations for x and label the solutions.



a. $-2 + \log x = -2$

$x =$ ____________
Label the solution with an A on the graph.

b. $-2 + \log x = -1$

$x =$ ____________
Label the solution with a B on the graph.

c. $-4 = -2 + \log x$

$x =$ ________
Label the solution with a C on the graph.

d. $-1.3 = -2 + \log x$

$x =$ ________
Label the solution with a D on the graph.

6. Are the solutions that you found in #5 exact or approximate? Why?

Equation Puzzles:
Solve each equation for *x.* Show exact solutions and decimal approximations.

7. $10^x = 10{,}000$

8. $125 = 10^x$

9. $-(10^{x+2}) = 16$

10. $5(10^{x+2}) = 0.25$

11. $10^{-x-1} = \frac{1}{36}$

12. $10^{x+2} = 347$

2.8 Warm Up
Exponential Equations

Solve each exponential equation by first changing the equation into logarithmic form and then use the change of base formula.

1. $2^{x+7} = 5$

2. $4 \cdot 3^{x} - 2 = 14$

3. $5^{x-2} + 1 = 18$

Solve each logarithmic equation by first changing the equation into exponential form

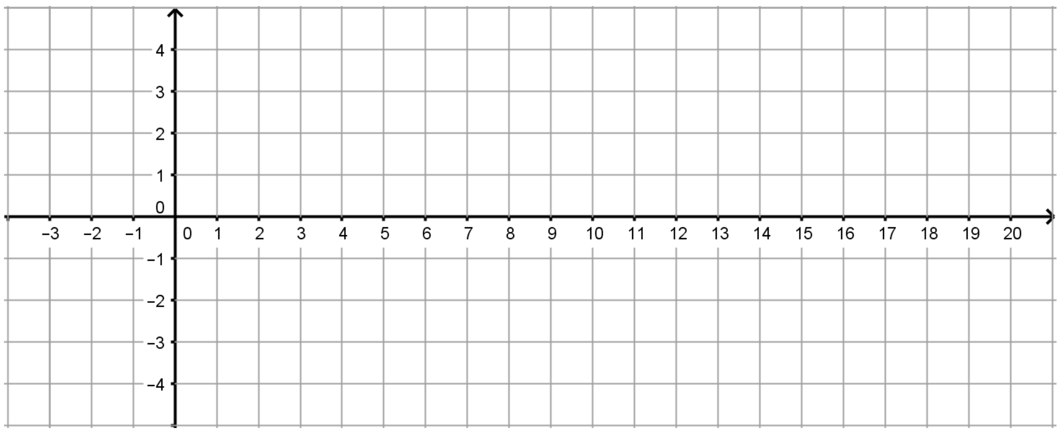
4. $2\log_2(x+1) = 2$

5. $\log(3x-3) = \log(x+1) + \log 4$

2.8 Log Rolling
A Practice Understanding Task

1. Consider the function, $f(x) = \log_3 9x$.
   a. Use properties of logarithms to show that the graph of $f(x)$ is a vertical transformation of the graph of $y = \log_3 x$.

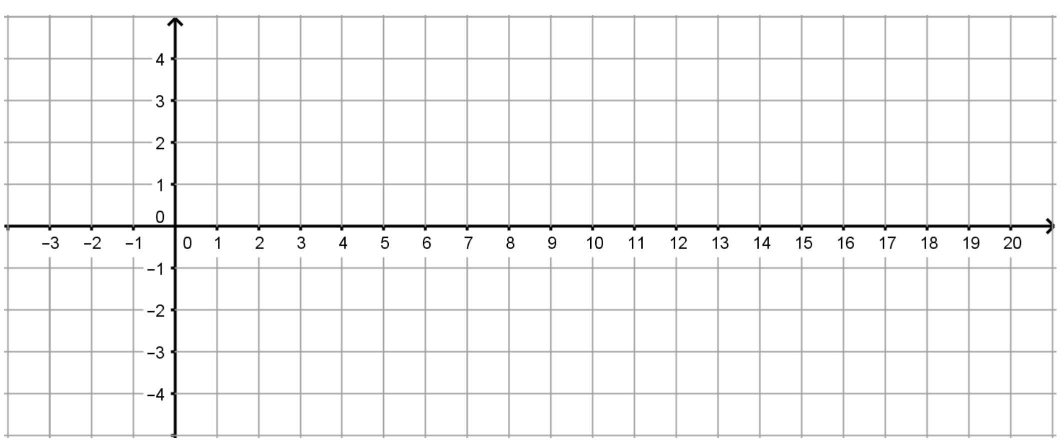
   b. Graph both $y = \log_3 x$ and $f(x) = \log_3 9x$ on the same set of axes below.



   c. Solve the equation $\log_3 9x = 4$. How can you use the graph of $f(x) = \log_3 9x$ to solve the equation?

2. Consider the function, $g(x) = \log_4 x^2 - \log_4 x$.
   a. Use properties of logarithms to express $g(x)$ as a single logarithm.

   b. Graph $g(x)$ below.



   c. Solve the equation $\log_4 x^2 - \log_4 x = 1.5$. How can you use the graph of $g(x)$ to solve the equation?

Integrated Math 3
Module 2
Logarithmic Functions
Ready, Set, Go! Homework

Adapted from

The Mathematics Vision Project:
Scott Hendrickson, Joleigh Honey, Barbara Kuehl,
Travis Lemon, Janet Sutorius

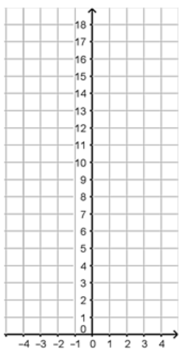
Ready, Set, Go!

Ready

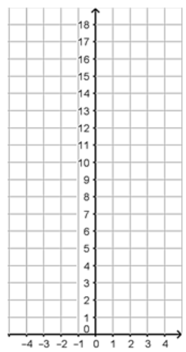
Topic: Graphing exponential functions

Graph each function over the domain $\{-4 \leq x \leq 4\}$. Reminder: $f(x) = b^{x-h} + k$ has a horizontal translation h units and a vertical translation k units.

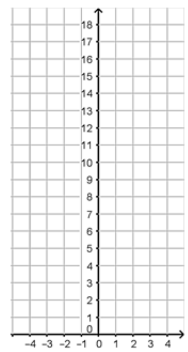
1. $y = 2^x$



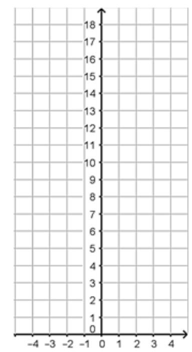
2. $y = 2 \cdot 2^x$



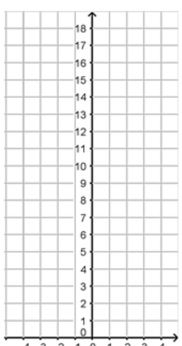
3. $y = \left(\frac{1}{2}\right)^x$



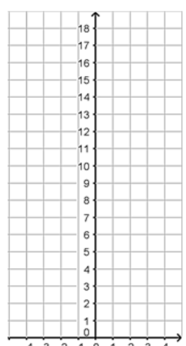
4. $y = 2\left(\frac{1}{2}\right)^x$



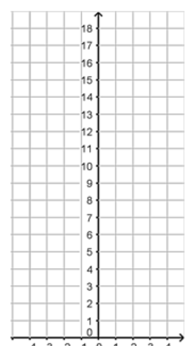
5. $y = 2^{x+1}$



6. $y = 2^x + 1$



7. $y = 2^{-x}$



8. Compare graph #1 to graph #2. Multiplying by 2 should generate a dilation of the graph, but the graph looks like it has been translated horizontally. How do you explain that? Hint: for question 2, use properties of exponents to simplify the function to have only one base.

9. Compare graph #3 to graph #4. Is your explanation in #8 still valid for these two graphs? Explain.

Set

Topic: Evaluating logarithmic functions

Arrange the following expressions in numerical order from smallest to largest. Do not use a calculator. Be prepared to explain your logic. Note: $\log x$ is the same as $\log_{10} x$.

	A	B	C	D	E
10.	$\log_2 32$	$\log_7 343$	$\log_{35} 1$	$\log_{15} 225$	$\log_{11} 11$
11.	$\log_3 81$	$\log_5 125$	$\log_8 8$	$\log_4 1$	$\log 100$
12.	$\log_{\frac{1}{2}} \frac{1}{8}$	$\log_{\frac{1}{3}} 9$	$\log_{\frac{1}{4}} \frac{1}{2}$	$\log_{\frac{1}{5}} 10$	$\log_x x$
13.	$\log_x \frac{1}{x^2}$	$\log_5 \frac{1}{5}$	$\log_2 \frac{1}{8}$	$\log \frac{1}{10{,}000}$	$\log_x 1$
14.	$\log 200$	$\log 0.02$	$\log_2 10$	$\log_2 \frac{1}{10}$	$\log_2 200$

Answer the following questions. If yes, give an example of the answer. If no, explain why not.

15. Is it possible for a logarithm to equal a negative number?

16. Is it possible for a logarithm to equal zero?

17. Does $\log_x 0$ have an answer?

18. Does $\log_x 1$ have an answer?

19. Does $\log_x x^5$ have an answer?

Go

Topic: Properties of exponents

Write each expression as an integer or a simple fraction.

20. $11(-6)^0$

21. -3^{-2}

22. $\frac{4^3}{8^0}$

23. $\frac{4^0}{2^{-5}}$

24. $\frac{9x^{\frac{1}{3}}}{2^{-1}x^{\frac{5}{3}}}$

25. $\left(42x^{\frac{1}{2}}\right)(6^{-4}x^2)$

26. $\left(2x^{\frac{3}{4}}\right)^2$

27. $\frac{7^{-2}x^7y^4}{4^{-1}x^{12}y^{-3}}$

28. $\frac{22^{-1}x^{-6}y^{-2}}{4^{-1}x^{-7}y^{-5}}$

Ready, Set, Go!

Ready
Topic: Solving simple logarithmic equations

Find the answer to each logarithmic equation. Then write each logarithmic equation as an exponential equation.

1. $\log_5 625 =$

2. $\log_3 243 =$

3. $\log_5 0.2 =$

4. $\log_9 81 =$

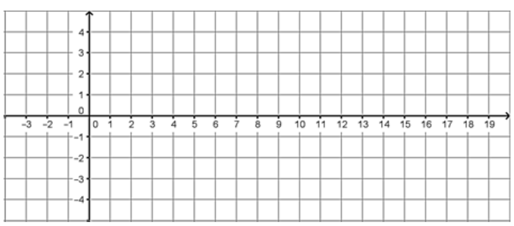
5. $\log 1{,}000{,}000 =$

6. $\log_x x^7 =$

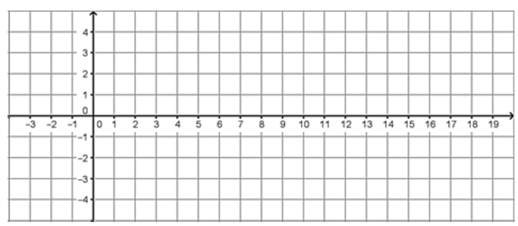
Set
Topic: Graphing logarithmic functions

Graph each logarithmic function below.

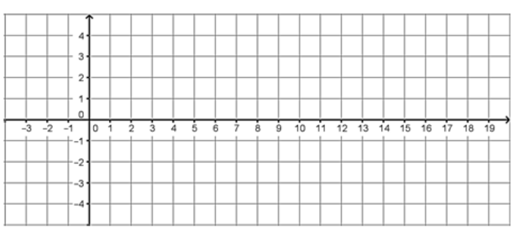
7. $f(x) = \log_3(x)$



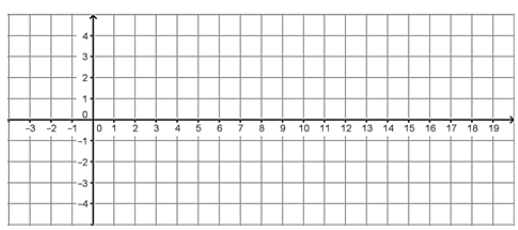
8. $g(x) = \log_3(x+3)$



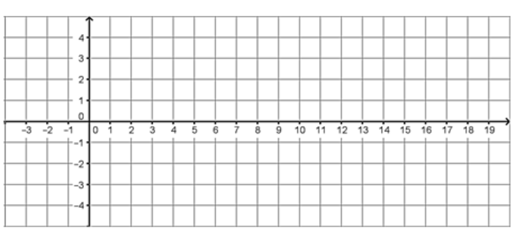
9. $f(x) = 2 + \log_3(x-1)$



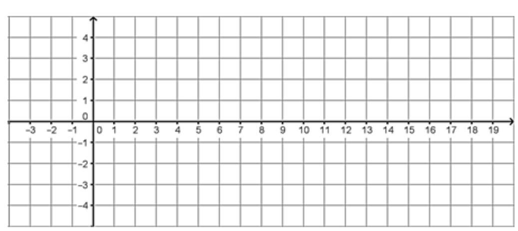
10. $g(x) = -1 + \log_3(x+2)$



11. $f(x) = 1 + \log_2(x+2)$



12 b. $g(x) = 3 \cdot \log_2(x+2)$



Topic: Evaluating logarithms.

Evaluate each logarithm without the use of a calculator.

13. $\log 10$

14. $\log_8 64$

15. $\log\left(\frac{1}{10}\right)$

16. $\log 1000$

17. $\log_2 32$

18. $\log_3\left(\frac{1}{27}\right)$

Go

Topic: Power to a power rule with exponents

Simplify each expression. Answers should have only positive exponents.

19. $(2^3)^4$

20. $(x^3)^2$

21. $(a^3)^{-2}$

22. $(b^{-7})^2$

23. $\left(\frac{24x^6y^3}{20x^3y^9}\right)\left(\frac{6x^7y^2}{30x^9y^6}\right)$

24. $(2^3w)^4$

Name **Logarithmic Functions | 2.3**

Ready, Set, Go!

Ready

Topic: Solving exponential equations

Solve each equation by first rewriting the equations so the base(s) are the same.

1. $4^x = 32$

2. $125^x = 625$

3. $5^{2-x} = \frac{1}{125}$

4. $6^{\frac{x-3}{4}} = \sqrt{6}$

5. $8^{x+3} = 16^{x-1}$

6. $4^x = \left(\sqrt{2}\right)^{x+1}$

7. $9^x = \frac{1}{\sqrt[3]{3}}$

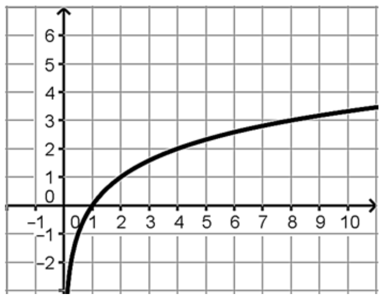
8. $4^{x+2} = 8^{1-x}$

Set

Topic: Transformations on logarithmic functions

Answer the questions about each graph.

9.



a. Find x when $f(x) = 0$?

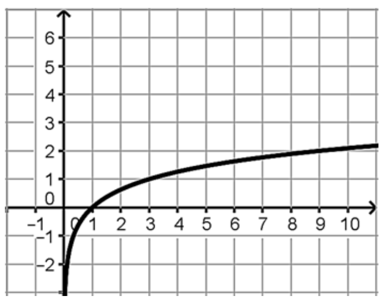
b. Find x when $f(x) = 1$?

c. Find $f(x)$ when $x = 2$?

d. Find x when $f(x) = 3$?

e. What is the equation of this graph?

10.



a. Find x when $f(x) = 0$?

b. Find x when $f(x) = 1$?

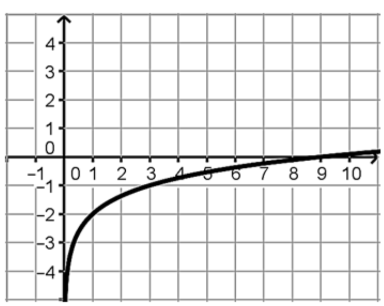
c. Find $f(x)$ when $x = 9$?

d. Find x when $f(x) = 2$?

e. What is the equation of this graph?

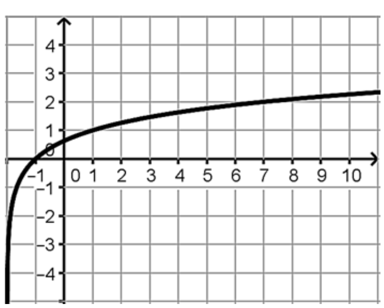
Use the graph and the table of values for the graph to write the equation of the graph. Explain which numbers in the table helped you the most to write the equation.

11.



x	y
1	-2
2	-1.369
3	-1
	-0.535
7	$-.2288$
9	0

12.



x	y
-2	Undefine
-1	0
0	0.63093
1	1
5	1.7712
6	1. 9 8
7	2

Go

Topic: Power to a power rule with exponents

Simplify each expression. Answers should have only positive exponents.

13. $x^2 \cdot (x^5)^2$

14. $m^{-3} \cdot (m^2)^3$

15. $\left(\frac{2a^3b^2}{18a^7b^{-4}}\right)^2$

16. $\left(x^{\frac{1}{3}}\right)^4 \cdot x^{\frac{1}{6}}$

17. $(5a^3)^2$

18. $(6x^7b^{-4})^0$

Name **Logarithmic Functions | 2.4**

Ready, Set, Go!

Ready

Topic: Fractional exponents

Write the following expressions with an exponent. Simplify when possible.

1. $\sqrt[5]{x}$
2. $\sqrt[3]{w^8}$
3. $\sqrt[5]{125m^5}$
4. $\sqrt[3]{9b^8}$

Rewrite each expression with a fractional exponent. Then find the value of the expression.

5. $\log_3 \sqrt[5]{3}$
6. $\log_7 \sqrt[5]{343}$

Set

Topic: Expanding logarithmic expressions

Use the properties of logarithms to expand the expression as a sum of logarithms. Simplify your result wherever possible. (Assume all variables are positive.)

7. $\log_5 7x$
8. $\log_5 10a$
9. $\log xyz$
10. $\log_3(27x)$
11. $\log_2\left(\frac{1}{4}xy\right)$
12. $\log_{25}(5a)$
13. $\log(100x)$
14. $\log\left(\frac{1}{1000}y\right)$
15. $\log_4(16xyz)$

Go

Topic: Writing expressions in exponential form and logarithmic form

Convert to logarithmic form.

16. $2^9 = 512$

17. $10^{-2} = 0.01$

18. $\left(\frac{2}{3}\right)^{-1} = \frac{3}{2}$

Write in exponential form.

19. $\log_4 2 = \frac{1}{2}$

20. $\log_{\frac{1}{3}} 3 = -1$

21. $\log_{\frac{2}{5}} \frac{8}{125} = 3$

Name **Logarithmic Functions | 2.5**

Ready, Set, Go!

Ready

Topic: Fractional exponents

Write the following expressions with an exponent. Simplify when possible.

1. $\sqrt[7]{s^2}$
2. $\sqrt[3]{8r^6}$
3. $\sqrt[3]{(8x)^2}$
4. $\sqrt{75x^6}$

Rewrite each expression with a fractional exponent. Then find the value of the expression.

5. $\log_2 \sqrt[3]{4}$
6. $\log_5 \sqrt[5]{3125}$

Set

Topic: Expanding logarithmic expressions

Use the properties of logarithms to expand the expression as a sum or difference, and/or constant multiple of logarithms. (Assume all variables are positive.)

7. $\log_5 \frac{d}{4}$
8. $\log_6 x^3$
9. $\log_5 9x^2$
10. $\log_2 (7x)^4$
11. $\log_3 \sqrt{w}$
12. $\log_5 \frac{xyz}{w}$
13. $\log_5 \frac{9\sqrt{x}}{y^3}$
14. $\log_2 \left(\frac{x^2-4}{x^3}\right)$
15. $\log_2 \left(\frac{x^2}{y^5 w^3}\right)$
16. $\log \left(\frac{xy}{100z^3}\right)$

Topic: Condensing logarithmic expressions

Use the properties of logarithms to condense the expression into a single logarithm. (Assume all variables are positive.)

17. $\log_5 3 - \log_5 x$

18. $3\log_6 x + 4\log_6 y$

19. $\log_2 a + 3\log_2 x$

20. $\log_2 x + \log_2 y - 4 - 5\log_2 z$

21. $\log_2(x+6) + \log_2(x-6) - 5\log_2 y$

Go

Topic: Writing expressions in exponential form and logarithmic form

Convert to logarithmic form.

22. $3^7 = 2187$

23. $10^{-4} = 0.0001$

Write in exponential form.

24. $\log_8 2 = \frac{1}{3}$

25. $\log_{\frac{1}{3}} 9 = -2$

Name **Logarithmic Functions | 2.6**

Ready, Set, Go!

Ready

Topic: Solving simple exponential and logarithmic equations

You have solved exponential equations before based on the idea that
$a^x = a^y$, ***if and only if*** $x = y$.

You can use the same logic on logarithmic equations. $\log_a x = \log_a y$, ***if and only if*** $x = y$.

Rewrite each equation so that the bases are the same on both sides. Then solve for *x*.

Example: Original equation	Rewritten equation:	Solution:
a. $3^x = 81$	$3^x = 3^4$	$x = 4$
b. $\log_2 x - \log_2 5 = 0$	$\log_2 x = \log_2 5$	$x = 5$

1. $3^{x+4} = 243$

2. $\left(\frac{1}{2}\right)^x = 8$

3. $\left(\frac{3}{4}\right)^x = \frac{27}{64}$

4. $\log_2 x - \log_2 13 = 0$

5. $\log_2(2x - 4) - \log_2 8 = 0$

6. $\log_2(x + 2) - \log_2 9x = 0$

7. $\frac{\log 2x}{\log 14} = 1$

8. $\frac{\log(5x-1)}{\log 29} = 1$

9. $\frac{\log 5^{(x-2)}}{\log 625} = 1$

Set

Topic: Rewriting logs in terms of known logs

Use the given values and the properties of logarithms to find the indicated logarithm. Do not use a calculator to evaluate the logarithms.

Given: $\mathbf{\log 16 \approx 1.2}$
$\mathbf{\log 5 \approx 0.7}$
$\mathbf{\log 8 \approx 0.9}$

10. Find $\log \frac{5}{8}$

11. Find $\log 25$

12. Find $\log \frac{1}{2}$

13. Find $\log 80$

14. Find $\log \frac{1}{64}$

Given: $\mathbf{\log_3 2 \approx 0.6}$
$\mathbf{\log_3 5 \approx 1.5}$

15. Find $\log_3 16$

16. Find $\log_3 100$

17. Find $\log_3 \frac{3}{50}$

18. Find $\log_3 \frac{8}{15}$

19. Find $\log_3 486$

20. Find $\log_3 18$

21. Find $\log_3 120$

22. Find $\log_3 \frac{32}{45}$

Go

Topic: Using the definition of logarithm to solve for *x*.

Use your calculator and the definition of log x (recall: $log\ x = log_{10}x$) to find the value of *x*. You may want to reqrite each in exponential form first. Round your answers to 4 decimals.

23. $\log x = -3$

24. $\log x = 1$

25. $\log x = 0$

26. $\log x = \frac{1}{2}$

27. $\log x = 1.75$

28. $\log x = -2.2$

29. $\log x = 3.67$

30. $\log x = \frac{3}{4}$

31. $\log x = 6$

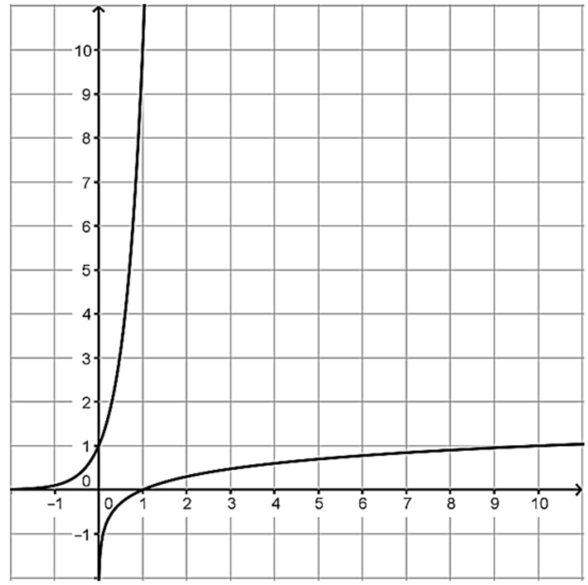
Name **Logarithmic Functions | 2.7**

Ready, Set, Go!

Ready

Topic: Comparing the exponential and logarithmic graphs

The graphs of $f(x) = 10^x$ and $g(x) = \log x$ are shown in the same coordinate plane. Make a list of the characteristics of each function. (Domain/range, increase/decrease, asymptotes, etc.)



1. $f(x) = 10^x$

2. $g(x) = \log x$

Each question below refers to the graphs of the functions $f(x) = 10^x$ and $g(x) = \log x$. State whether each statement is true or false. If they are false, correct the statement so that it is true.

__________ 3. Every graph of the form $g(x) = \log_b x$ will contain the point $(1, 0)$.

__________ 4. Both graphs have vertical asymptotes.

__________ 5. The graphs of $f(x)$ and $g(x)$ have the same rate of change.

__________ 6. The functions are inverses of each other.

__________ 7. The range of $f(x)$ is the domain of $g(x)$.

__________ 8. The graph of $g(x)$ will never reach 3.

Set

Topic: Using logarithmic properties to evaluate logs

Evaluate the following logarithms.

9. $\log 10$
10. $\log 10^{-7}$
11. $\log 10^{75}$
12. $\log 10^{x}$

13. $\log_3 3^5$
14. $\log_8 8^{-3}$
15. $\log_{11} 11^{37}$
16. $\log_m m^n$

You can use this property illustrated above to help you solve logarithmic equations. *Note that this property only works when the base of the logarithm matches the base of the exponent.*

Topic: Solving logarithmic equations taking the log of each side.

Solve the equations by using a property of logarithms on both sides of the equation. You will need a calculator. Round solutions to 3 decimal places.

17. $10^n = 4.305$
18. $5^n = 0.316$
19. $2^n = 14{,}521$
20. $10^n = 483.059$

Go

Topic: Expanding and condensing logarithms

Expand each logarithm.

21. $\log_3 \frac{243x^7y^4}{8z^5}$

22. $\log_4 \frac{64p^5}{w^3b^8}$

Condense each logarithm.

23. $3\log_2 w - 5\log_2 t - \log_2 m$

24. $\frac{1}{2}\log_3 y - 4\log_3 q + 5\log_3 z + \log_3 h$

Ready, Set

Ready

Topic: Identifying domain

The following problems are intended to get you ready for the next module. Identify the domain of each function.

1. $f(x) = \sqrt{x-6}$

2. $f(x) = 2(x+4)^2 - 7$

3. $f(x) = 5 + \log_2(x+3)$

4. $f(x) = 2^{x-1} - 8$

5. $f(x) = \frac{1}{x-4}$

Set

Topic: Expanding and condensing logarithmic expressions.

Expand each logarithm.

6. $\log_4 \frac{64x^6y^8}{z^2}$

7. $\log_2 \sqrt{x-4}$

8. $\log_5 \frac{125x^4\sqrt[3]{y^2}}{z^7\sqrt{p}}$

Condense each logarithm.

9. $2\log_3 x - 4\log_3 z - 6\log_3 y$

10. $\frac{1}{2}\log_4 w + 5\log_4 p - 3\log_4 t$

11. $7\log_x 4 + 2\log_x y - 4\log_x z$

Topic: Solving logarithmic equations

Solve each logarithmic equation.

12. $\log_4(2x+8) = \log_4(6x-12)$

13. $\log(x-3) = \log(7x-23) - \log(x+1)$

14. $\log_4(2x+1) = \log_4(x-3) + \log_4(x+5)$

Solve each logarithmic equation using the definition of a logarithm.

15. $\log x = 6$

16. $\log_2 x = 8$

17. $\log_3 x = -2$

Topic: Solving exponential equations

Solve each exponential equation.

18. $2^{4x-2} = 64$

19. $9^{x+2} = 27^{-x}$

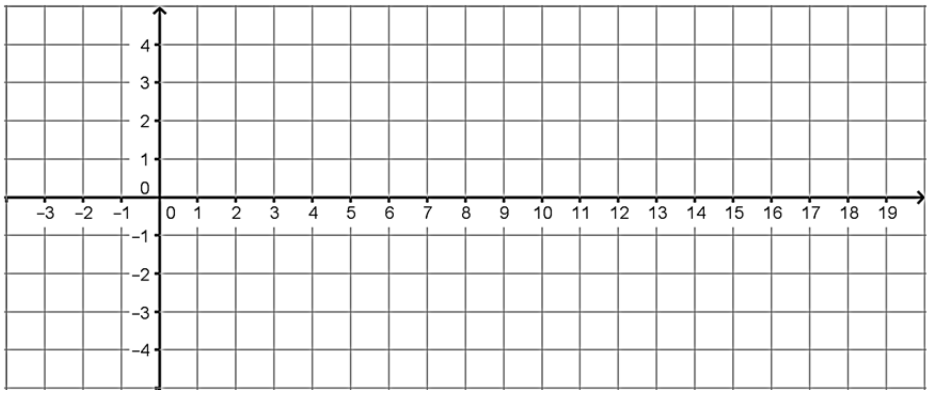
20. $10^x = 15$

21. $10^{x+4} = 200$

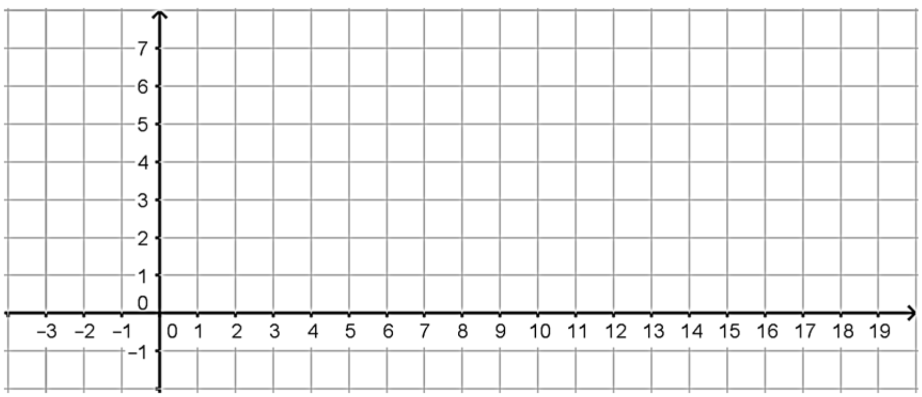
Topic: Graphing logarithmic functions.

Graph each logarithmic function.

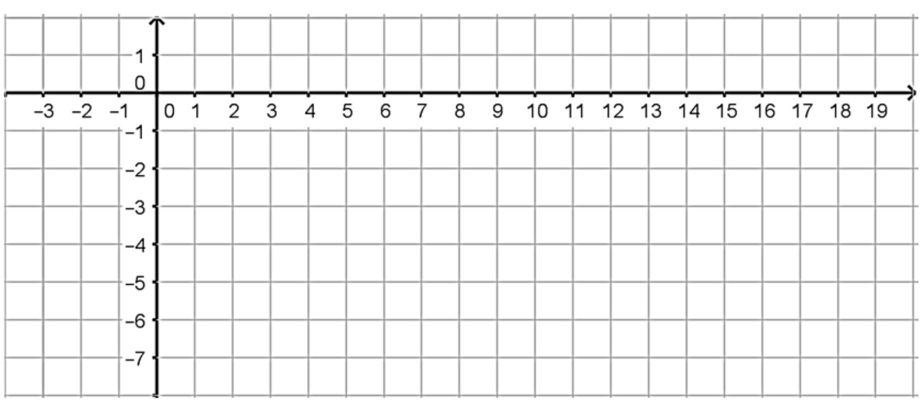
22. $f(x) = \log_{\frac{1}{2}}(x+3)$



23. $f(x) = 2 + \log_2 x$



24. $f(x) = -3 + \log_3(x-4)$



Integrated Math 3
Module 3
Polynomial Functions

Adapted from

The Mathematics Vision Project:
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Travis Lemon, Janet Sutorius

Module 3 Overview

Prerequisite Concepts and Skills:

- Factoring quadratic expressions
- Solving quadratic equations
- Features of linear, exponential, radical, and quadratic functions (increasing, decreasing, domain, range, intercepts, min/max)
- Add/subtract/multiply polynomial expressions
- Fundamental Theorem of Algebra (for linear and quadratic)
- Complex roots

Summary of the Concepts & Skills in Module 3:

- Introduce cubic functions and polynomial functions of higher degree
- Understand end behavior of linear, exponential, radical, and polynomial functions
- Add, subtract, and multiply polynomial functions using a graphical representation
- Binomial expansion using Pascal's Triangle
- Applying the Fundamental Theorem of Algebra for polynomial functions of any degree
- Use the Remainder Theorem to find linear factors and roots of polynomial functions
- Division of polynomial functions by a linear factor

Content Standards and Standards for Mathematical Practice Covered:

- Content Standards: F.BF.1, F.LE.3, F.IF.4, F.IF.4, F.IF.7, F.IF.9, A.SSE.1, A.APR.1, A.APR.2, A.APR.3, A.APR.5, N.CN.8, N.CN.9
- Standards for Mathematical Practice:
  1. Make sense of problems and persevere in solving them.
  2. Reason abstractly and quantitatively.
  3. Construct viable arguments and critique the reasoning of others.
  4. Model with mathematics.
  5. Use appropriate tools strategically.
  6. Attend to precision.
  7. Look for and make use of structure.
  8. Look for and express regularity in repeated reasoning.

Module 3 Vocabulary:

- Cubic function
- End behavior as $x \to \infty$
- Roots
- Pascal's Triangle
- Remainder Theorem
- Irrational roots
- Magnitude of functions
- End behavior as $x \to -\infty$
- Binomial expansion
- Fundamental Theorem of Algebra
- Complex roots

Concepts Used In the Next Module:

- Features of functions
- Polynomial division
- End behavior
- Finding roots

Module 3 – Polynomial Functions

3.1 *Comparing growth rates of linear, quadratic, and cubic functions and recognizing that cubic functions can be created from the sums of the terms of a quadratic sequence* **(F.BF.1, F.LE.3)**
Warm Up: Linear and Quadratic Patterns
Classroom Task: Scott's Macho March Madness – A Develop Understanding Task
Ready, Set, Go Homework: Polynomial Functions 3.1

3.2 *Determining the slowest to the fastest growing functions by ordering and comparing values as x approaches infinity* **(F.LE.3, A.SSE.1, F.IF.4)**
Warm Up: Boxing Day
Classroom Task: Which is Greater? – A Develop Understanding Task
Ready, Set, Go Homework: Polynomial Functions 3.2

3.3 *Understanding end behavior and comparing end behavior of functions in different representations* **(F.IF.6, F.IF.7, F.IF.9)**
Warm Up: Graphing Function Operations
Classroom Task: All About Behavior – A Practice Understanding Task
Ready, Set, Go Homework: Polynomial Functions 3.3

3.4 *Using graphical representations to add, subtract, and multiply polynomials* **(A.APR.1, F.BF.1)**
Warm Up: This is the End
Classroom Task: Polynomial Connections – A Solidify Understanding Task
Ready, Set, Go Homework: Polynomial Functions 3.4

3.5 *Determining the nature of roots and applying the Fundamental Theorem of Algebra.* **(A.APR.3, N.CN.9)**
Classroom Task: The Expansion – A Develop Understanding Task
Ready, Set, Go Homework: Polynomial Functions 3.5

3.6 *Review of multiplying polynomials using the box method and dividing polynomials using the box method* **(A.APR.1)**
Warm Up: Boxing it Up
Classroom Task: UnBoxing Polynomials – A Practice Understanding Task
Ready, Set, Go Homework: Polynomial Functions 3.6

3.7 *Applying the Fundamental Theorem of Algebra* **(N.CN.8, N.CN.9, A.APR.3, F.IF.4, F.IF.7)**
Warm Up: Bigger Boxes
Classroom Task: Seeing Structure – A Solidify Understanding Task
Ready, Set, Go Homework: Polynomial Functions 3.7

3.8 *Using the Remainder Theorem to find all linear factors and roots of a polynomial function* **(N.CN.8, N.CN.9, A.APR.2, A.APR.3, F.IF.7)**
Warm Up: Boxing Match
Classroom Task: Graphing All Poly's – A Solidify Understanding Task
Ready, Set, Go Homework: Polynomial Functions 3.8

3.9 *Using properties of polynomial functions to write equations* **(N.CN.8, N.CN.9, A.APR.2, A.APR.3)**
Warm Up: It's All in Your Imagination
Classroom Task: Calling All Poly's – A Solidify Understanding Task
Ready, Set, Go Homework: Polynomial Functions 3.9

3.10 *Practicing all things related to graphing and solving polynomial functions* **(N.CN.9, A.APR.2, A.APR.3, F.IF.7)**
Warm Up: Begin Task
Classroom Task: I Know, What Do You Know? – A Practice Understanding Task
Ready, Set, Go Homework: Polynomial Functions 3.10

3.1 Warm Up
Linear and Quadratic Patterns

Use the first and second differences to determine if each pattern is linear or quadratic. Then write the explicit and recursive rules for each pattern.

1.

n	$f(n)$
1	5
2	3
3	1
4	-1
5	-3

Type of Pattern:

Recursive Rule:

Explicit Rule:

2.

n	$f(n)$
1	-3
2	0
3	5
4	12
5	21

Type of Pattern:

Recursive Rule:

Explicit Rule:

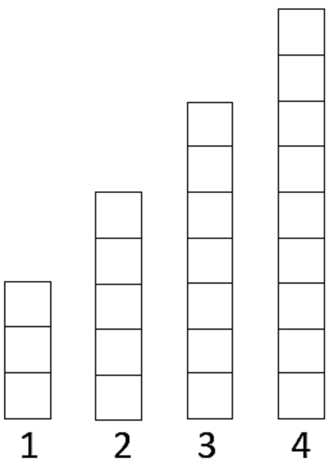
3.1 Scott's Macho March Madness
A Develop Understanding Task

http://www.flickr.com/photos/whyld/2282216226/

Each year, Scott participates in the "Macho March" promotion. The goal of "Macho March" is to raise money for charity by finding sponsors to donate based on the number of push-ups completed within the month. Last year, Scott was proud of the money he raised, but was also determined to increase the number of push-ups he would complete this year.

Part 1: Revisiting the Past

Below is the bar graph and table Scott used last year to keep track of the number of push-ups he completed each day, with day one showing he completed three push-ups and day two showing he completed five push-ups (for a total of eight completed push-ups at the end of day two). Scott continued the pattern seen in the bar graph throughout the month.



n Day of Month	$f(n)$ Push-ups **each day**	$g(n)$ **Cumulative** number of push-ups completed for the month
1	3	3
2	5	8
3	7	15
4	9	24
5	11	35
…	…	
n		

1. Write the recursive and explicit equations for the number of push-ups Scott completed **on any** given day last year. Explain how your equations connect to the bar graph and the table above.

2. Write the recursive and explicit equation for the **cumulative number** of push-ups Scott completed on any given day during the "Macho March" promotion last year.

Part 2: Macho March Madness

This year, Scott's plan is to look at the total number of push-ups he *completed for the month* on any given day last year, and do that many push-ups on the same day this year. For example, on day one, he will do three push-ups. On day two, he will do eight push-ups (the sum or total number of push-ups he completed on day one and two from last year). On day three, he will complete________push-ups. If Scott follows this pattern, determine the following:

Use the results from the questions below to fill in the chart on the resource page

3. How many push-ups will Scott complete on day five? How did you come up with this number? Write the recursive equation to represent the number of push-ups Scott will complete today based on the number of push-ups he completed yesterday.

4. How many **cumulative** push-ups will Scott have completed for the month on day five?

5. Without finding the explicit equation, make a conjecture as to the type of function that would represent the explicit equation for the cumulative number of push-ups Scott would complete on any given day this year. Hint: look at the consecutive differences within a table of values.

6. How does the rate of change for this explicit equation compare to the rates of change for the explicit equations in questions 1 and 2?

7. When looking at consecutive differences, how does the rate of change compare to the explicit equation for the function.

3.2 Warm Up
Boxing Day

Read through the example below to see how to multiply two polynomials using the “box method.” Then, use the box method to multiply the polynomials in questions 1-4.

Example: Multiply $(x^2 + 3x - 7)(2x^3 - 5x + 1)$

Set up a box whose length and height will accommodate the terms in each polynomial. In this case, we will use a 3×4 box (there will need to be an additional column for the $0x^2$ term in the second polynomial). One polynomial is listed along the left side; the other is listed across the top (be sure to include any negative signs). Use the idea of finding area = length $\times$ height to find the product that goes in to each cell.

	$2x^3$	$0x^2$	$-5x$	$+1$
x^2	$2x^5$	$0x^4$	$-5x^3$	x^2
$+3x$	$6x^4$	$0x^3$	$-15x^2$	$3x$
-7	$-14x^3$	$0x^2$	$35x$	-7

Once the cells are all filled, write the sum of all the terms within the cells and combine like terms.

$2x^5 + 6x^4 - 14x^3 - 5x^3 - 15x^2 + 35x + x^2 + 3x - 7$

$\mathbf{2x^5 + 6x^4 - 19x^3 - 14x^2 + 38x - 7}$

$(x^2 + 3x - 7)(2x^3 - 5x + 1) = \mathbf{2x^5 + 6x^4 - 19x^3 - 14x^2 + 38x - 7}$

Multiply the polynomials. Write the answer in standard form (descending order of powers).

1. $(x^2 + 3x - 1)(2x^2 - 5x - 1)$

2. $(x^3 - 5)(x^2 + 9x - 8)$

4. $(2x - 1)(x^4 - 3x^3 + x^2 - 5x + 12)$

5. $(2x^3 + 3x^2 - 4x + 2)(x^3 - 5x + 7)$

3.2 Which is Greater?
A Develop Understanding Task

In previous mathematics courses, you compared and analyzed growth rates of polynomial (mostly linear and quadratic) and exponential functions. In this task, we are going to analyze rates of change and end behavior by comparing various one variable expressions.

1. Compare and contrast linear, quadratic, and exponential functions.

2. a. Write the following functions in order from **least to greatest** when the value of x is zero.

   $A(x) = x^2 - 20$ $\quad$ $B(x) = x^5 - 4x^2 + 1$ $\quad$ $C(x) = x + 30$

   $D(x) = x^4 - 1$ $\quad$ $E(x) = x^3 + x^2 - 4$ $\quad$ $F(x) = -x^2 + 3x$

   b. Do you think this order would change when x represents other numbers?

3. Write the same functions in order from **least to greatest** when x represents a very large number (this number is so large, it is 'close to' or approaching positive infinity and is denoted as $x \to \infty$).

4. Write the same functions in order from **least to greatest** when x represents a number that is approaching negative infinity (as $x \to -\infty$).

5. When comparing functions, how does the order change depending on the values of x (close to negative infinity, zero, and close to positive infinity)? Hint: think about the degree of the function and where it is located in each list.

6. Determine where you would insert the following functions in your list from question 3. Then rewrite your list to contain all the functions.

   $G(x) = \left(\frac{1}{2}\right)^x$ $\quad H(x) = x^7$ $\quad I(x) = -x^5$ $\quad J(x) = x^6$ $\quad K(x) = 2^x$

7. Now insert these same functions to your list from question 4 and rewrite the order.

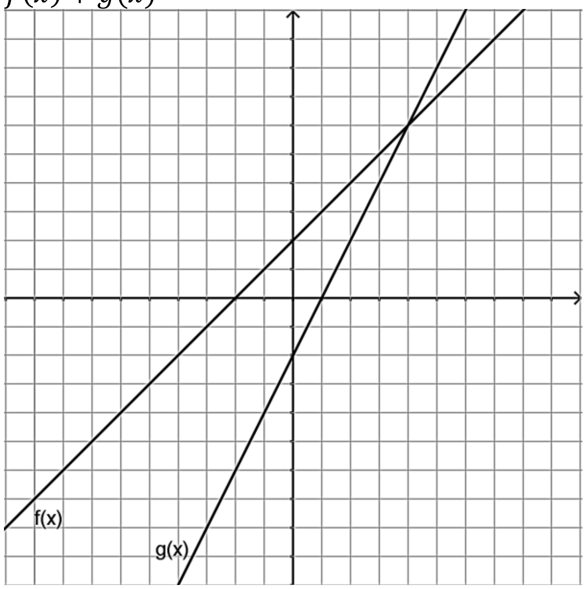
8. Write your process for ordering one variable polynomial functions for both extremes (when x approaches infinity as well as when x approaches negative infinity).

9. Discuss any other features of the functions you notice.

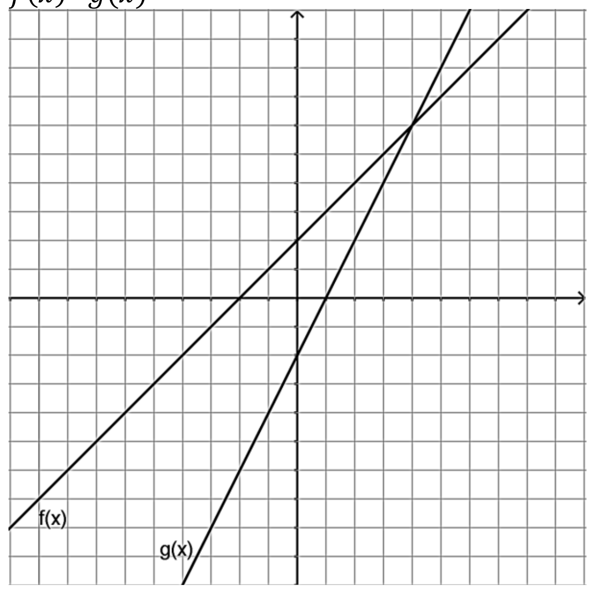
3.3 Warm Up
Graphing Function Operations

Use the graphs of $f(x)$, $g(x)$, and $h(x)$ to sketch the graphs of the following:

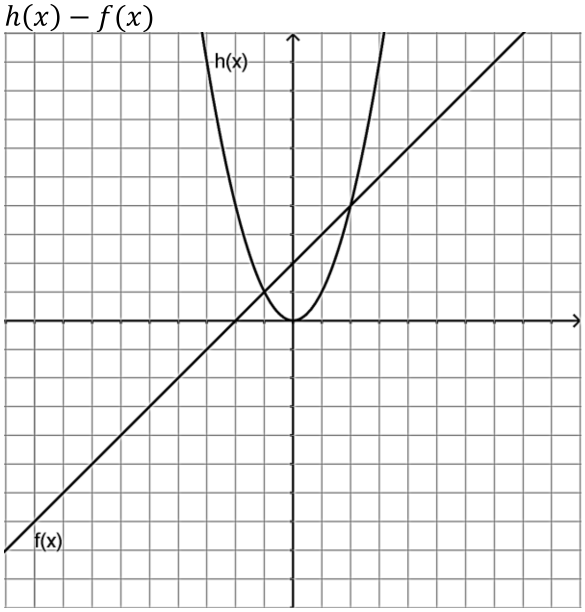
1. $f(x)+g(x)$



2. $f(x)\cdot g(x)$



3. $h(x)-f(x)$



4. Complete each sentence below.
   a. The sum or difference of two linear functions is…

   b. The product of two linear functions is…

   c. The sum or difference of a linear function and a quadratic function is…

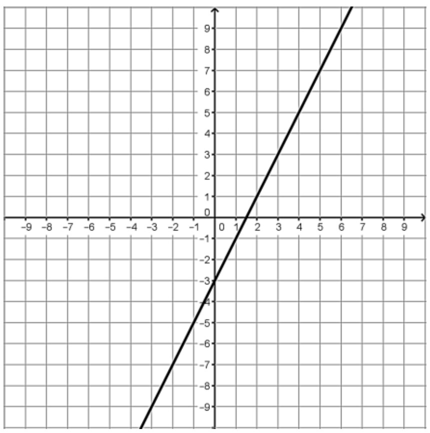
3.3 All About Behavior
A Practice Understanding Task

Part 1: For each situation:

- Determine the function type. If it is a polynomial, also state the degree of the polynomial and whether it is an even degree polynomial or an odd degree polynomial.
- For each, state the end behavior based on your knowledge of the function Use the format: As $x \to -\infty, f(x) \to$ _____ and as $x \to \infty, f(x) \to$ _____

Use the graphs below to determine the type of function, degree, if the degree is even or odd, and describe the end behavior of each function.

1.



Type of function:

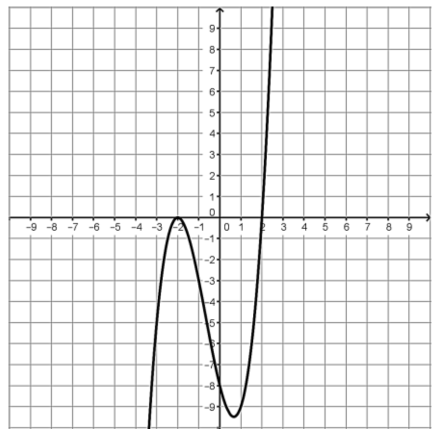
Least Degree:

Even or Odd Degree:

End Behavior:
as $x \to -\infty, f(x) \to$ _____

as $x \to \infty, f(x) \to$ _____

2.



Type of function:

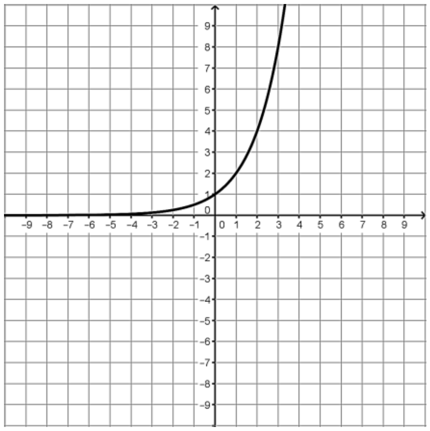
Least Degree:

Even or Odd Degree:

End Behavior:
as $x \to -\infty, f(x) \to$ _____

as $x \to \infty, f(x) \to$ _____

3.



Type of function:

Least Degree:

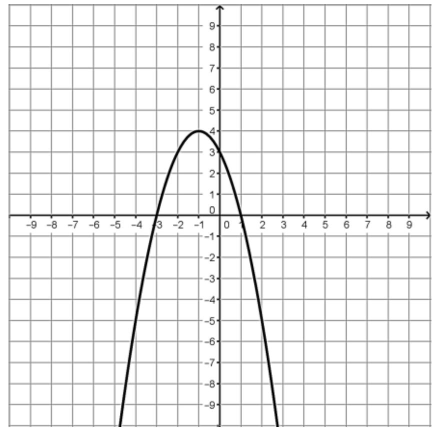
Even or Odd Degree:

End Behavior:

as $x \to -\infty, f(x) \to$ _____

as $x \to \infty, f(x) \to$ _____

4.



Type of function:

Least Degree:

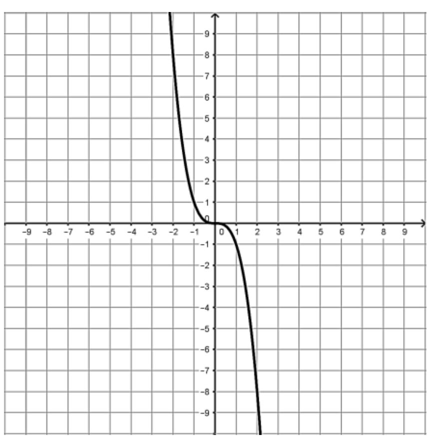
Even or Odd Degree:

End Behavior:

as $x \to -\infty, f(x) \to$ _____

as $x \to \infty, f(x) \to$ _____

5.



Type of function:

Least Degree:

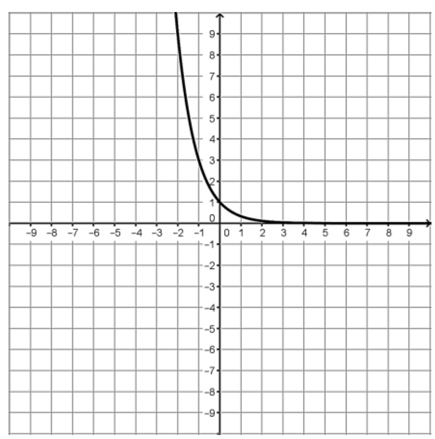
Even or Odd Degree:

End Behavior:

as $x \to -\infty, f(x) \to$ _____

as $x \to \infty, f(x) \to$ _____

6.



Type of function:

Least Degree:

Even or Odd Degree:

End Behavior:

as $x \to -\infty, f(x) \to$ _____

as $x \to \infty, f(x) \to$ _____

7. $f(x) = 3 + 2x$
Type of function:

Degree:

Even or Odd Degree:

End Behavior:

8. $f(x) = x^4 - 16$
Type of function:

Degree:

Even or Odd Degree:

End Behavior:

9. $f(x) = 3^x$
Type of function:

Degree:

Even or Odd Degree:

End Behavior:

10. $f(x) = x^3 + 2x^2 - x + 5$
Type of function:

Degree:

Even or Odd Degree:

End Behavior:

11. $f(x) = -2x^3 + 2x^2 - x + 5$
Type of function:

Degree:

Even or Odd Degree:

End Behavior:

12. $f(x) = \log_2 x$
Type of function:

Degree:

Even or Odd Degree:

End Behavior:

13. $f(x) = -2(x-3)(x+4)$
Type of function:

Degree:

Even or Odd Degree:

End Behavior:

14. $f(x) = \sqrt{x} - 3$
Type of function:

Degree:

Even or Odd Degree:

End Behavior:

15. $f(x) = 3(x-1)(x+2)(x-4)$
Type of function:

Degree:

Even or Odd Degree:

End Behavior:

Part 2: Use the functions from questions 1–15 to answer the following. Write a short explanation for each answer.

16. Compare questions 10 and 11: Which has the greatest value as $x \to \infty$?

17. Compare questions 3 and 12: Which has the greatest value as $x \to \infty$?

18. Compare questions 1 and 14: Which has the greatest value at as $x \to \infty$?

19. Compare questions 8 and 10: Which of these two polynomials has the highest degree?

20. Compare questions 4 and 13: Which has the highest maximum value?

21. Compare questions 10 and 14: Which has the greatest average rate of change over the interval $[15, 20]$?

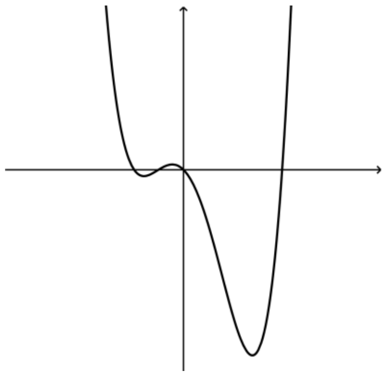
22. Compare questions 3 and 5: Which grows faster as $x \to \infty$?

23. Extension: Create three comparison questions of your own (be sure you know the answer).

3.4 Warm Up
This is the End

State whether the given graph is a polynomial, the least degree (if possible), whether the degree is even or odd, and the end behavior. Be sure to use correct notation (As $x \to \pm\infty, f(x) \to \pm\infty$).

a.



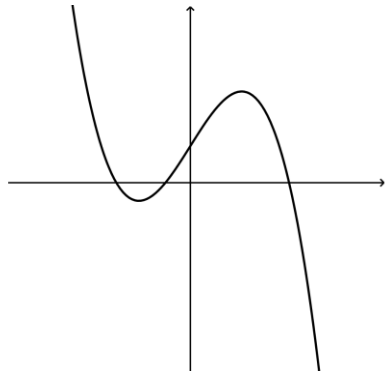
Type of function:

Least Degree:

Even or Odd Degree:

End Behavior:
as $x \to -\infty, f(x) \to$
as $x \to \infty, f(x) \to$

b.



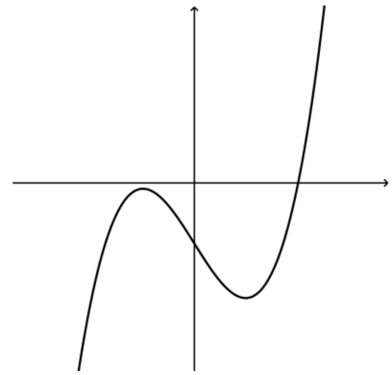
Type of function:

Least Degree:

Even or Odd Degree:

End Behavior:
as $x \to -\infty, f(x) \to$
as $x \to \infty, f(x) \to$

c.



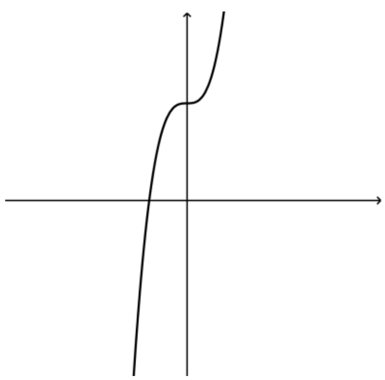
Type of function:

Least Degree:

Even or Odd Degree:

End Behavior:
as $x \to -\infty, f(x) \to$
as $x \to \infty, f(x) \to$

d.



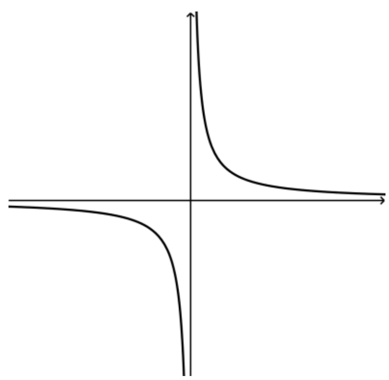
Type of function:

Least Degree:

Even or Odd Degree:

End Behavior:
as $x \to -\infty, f(x) \to$
as $x \to \infty, f(x) \to$

e.



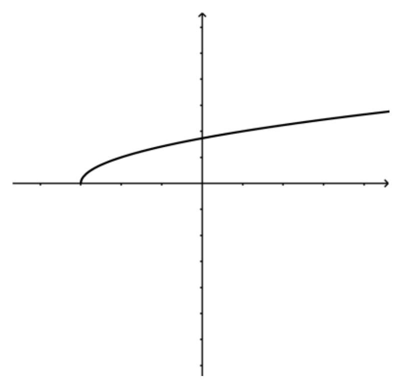
Type of function:

Least Degree:

Even or Odd Degree:

End Behavior:
as $x \to -\infty, f(x) \to$
as $x \to \infty, f(x) \to$

f.



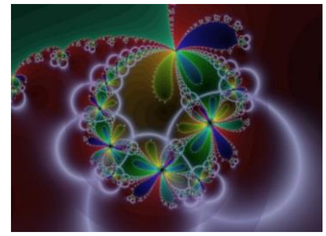
Type of function:

Least Degree:

Even or Odd Degree:

End Behavior:
as $x \to -3, f(x) \to$
as $x \to \infty, f(x) \to$

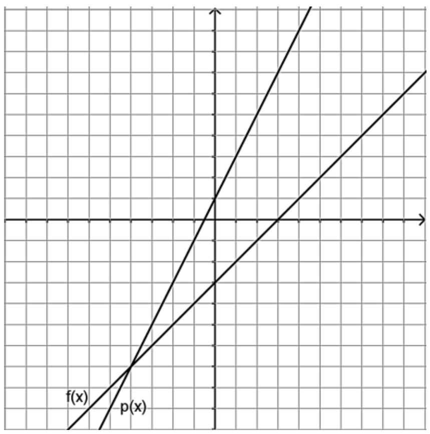
3.4 Polynomial Connections
A Solidify Understanding Task



This task is about using what we know to make conjectures about features of polynomial functions.

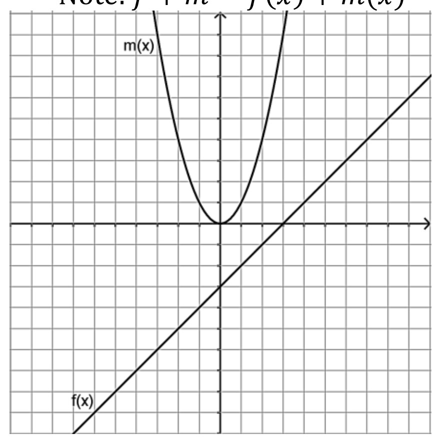
Use the graphs above to complete questions 1—6. On the same set of axes, graph the solution to each question. Record observations about the relationship(s) between the original functions and the new function.

1. $f(x) + p(x)$



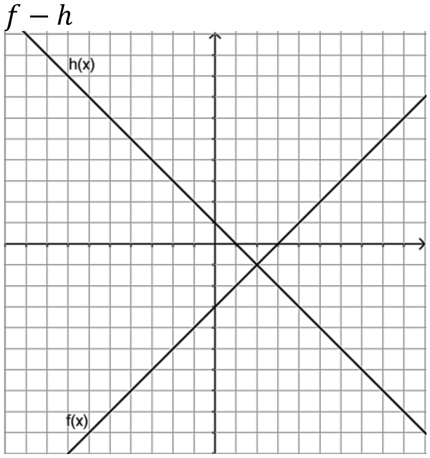
Observations:

2. $f + m$
   Note: $f + m = f(x) + m(x)$



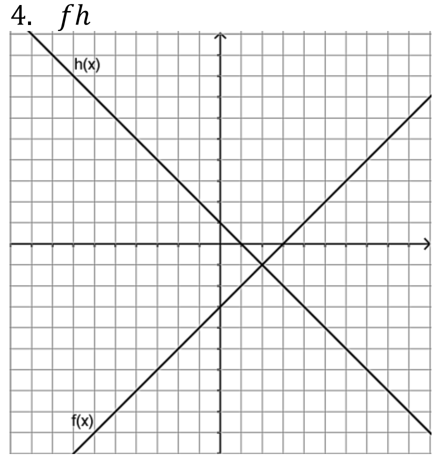
Observations:

3. $f - h$



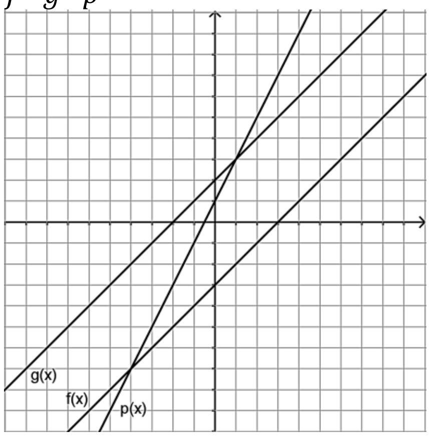
Observations:

4. fh



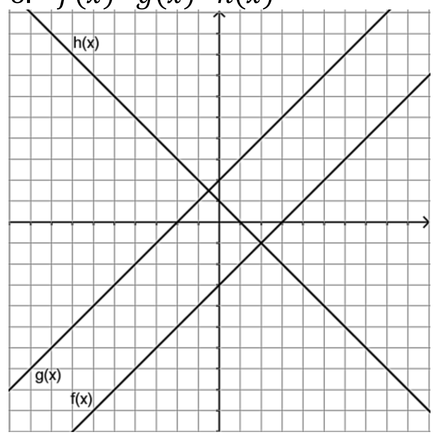
Observations:

5. $f \cdot g \cdot p$



Observations:

6. $f(x) \cdot g(x) \cdot h(x)$



Observations:

7. Describe your strategy for combining functions graphically. What methods did you seem to use more often?

8. Based on your experience in this task, describe the results when you add, subtract, or multiply linear functions. Make as many conjectures about the results of adding, subtracting, and multiplying linear factor equations and the resulting polynomials as possible.

3.5 The Expansion
A Develop Understanding Task

Polynomial functions have interesting characteristics. The degree of the polynomial not only tells us information about the end behavior of the function, it also tells us about the number of roots. This idea is called the **Fundamental Theorem of Algebra**.

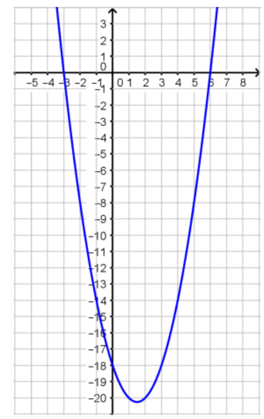
The theorem states:

***An n^{th} degree polynomial function has n roots*.**

You have seen how all linear functions have 1 root (in this case, it was the x-intercept), quadratic functions have 2 roots (these were sometimes the x-intercepts, but could also be imaginary). You found that a was a root of a function if $f(a) = 0$.

Here is an example:

$f(x) = x^2 - 3x - 18$ has roots $x = -3$ and $x = 6$ because $f(-3) = 0$ and $f(6) = 0$.
The graph at right illustrates the roots of $f(x)$:

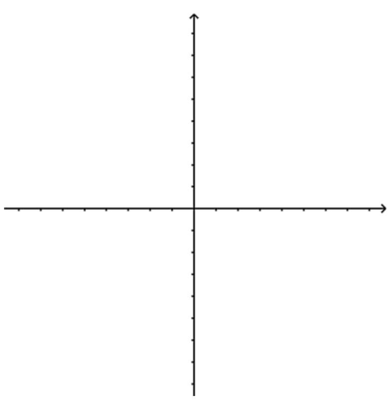


Part 1:

As we move onto polynomials with degree greater than two, let's see if The Fundamental Theorem of Algebra holds true for all polynomial functions. Make a conjecture as to the shape of each function and **sketch** this conjecture on the graph below. Note: in question 7, you will be testing your conjectures using technology.

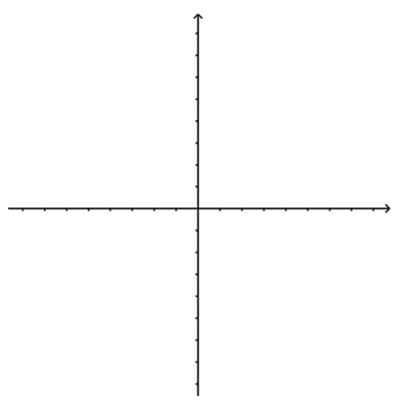
1. $f(x) = (x + 1)(x - 2)$

   Conjecture:



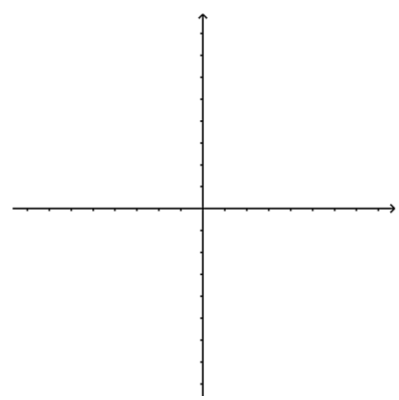
2. $f(x) = x(x + 1)^2$

   Conjecture:



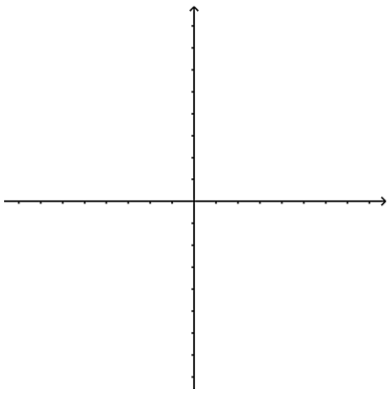
3. $f(x) = (x + 1)^2(x - 2)^2$

   Conjecture:



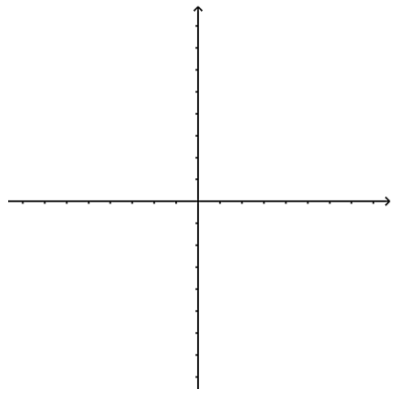
4. $f(x) = x + 1$

Conjecture:



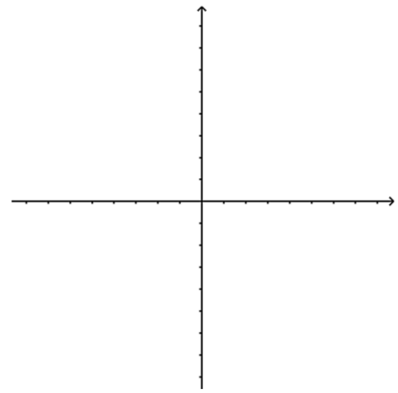
5. $f(x) = (x + 1)^2$

Conjecture:



6. $f(x) = (x + 1)^3$

Conjecture:



7. Once you have made a conjecture about the graphs, confirm your solutions (using technology) and **sketch** the graph in a **new color** on the same set of axes. How were your conjectures confirmed? What do you need to adjust?

8. Describe the number and types of roots of each function from questions 1-6. How does this match what you know about polynomials and the Fundamental Theorem of Algebra?

$f(x) = (x + 1)(x - 2)$

$f(x) = x(x + 1)^2$

$f(x) = (x + 1)^2(x - 2)^2$

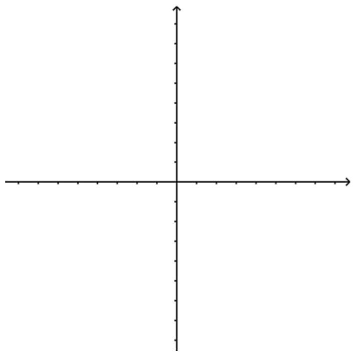
$f(x) = x + 1$

$f(x) = (x + 1)^2$

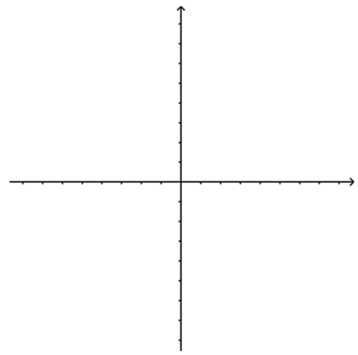
$f(x) = (x + 1)^3$

9. Without using technology, *sketch* the graph of each function. Be sure to accurately show what happens along the x-axis for each.

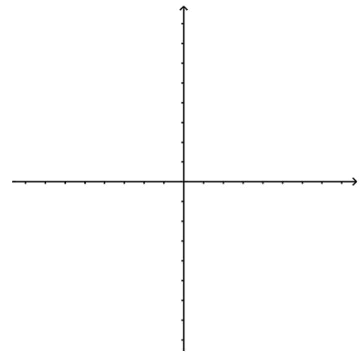
a. $f(x) = (x-3)^2(x+5)$



b. $g(x) = -\frac{1}{2}x(x+1)^3$

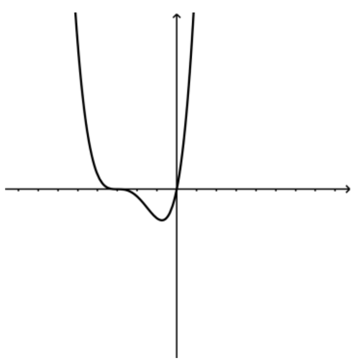


c. $h(x) = x(2x+1)^2(x-1)$

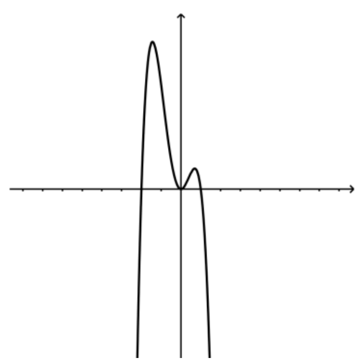


10. Find a possible equation for the given graphs.

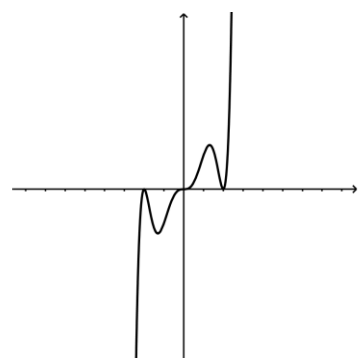
a.



b.



c.



3.6 Warm Up
Boxing it Up

Use the box method to multiply the polynomials below.

1. $(2x^2 + 3x - 1)(3x + 2)$

2. $(4x^2 - 5x + 1)(2x - 3)$

3.6 UnBoxing Polynomials
A Practice Understanding Task

https://www.vectorstock.com/royalty-free-vector/boxing-glove-out-of-the-box-vector-1468107

Rocky and Mickey were working on their math homework together and came across a problem that had them puzzled.

Rocky said "Yo, it gives us the answer here, we just need to find the question."

Mickey replied, "You're right! In previous problems, we knew the two factors and used a box to find the product. If we can use a box to multiply, we should be able to use a box to find out the factor."

"But how?" asked Rocky. "You might have to coach me through this one."

Here's the problem they were working on:

The product of two polynomials is $\boldsymbol{x^3+7x^2+7x-15}$. *One of the factors is* $\boldsymbol{x+3}$. *Find the other factor.*

Mickey drew the box below and placed $\boldsymbol{x^3}$ and $\boldsymbol{-15}$ in the box as shown.

$\boldsymbol{x^3}$		
		$\boldsymbol{-15}$

1. How did Mickey know that these terms were in the correct location?

2. He then filled in some terms along the outer edge of the box:

	x^2		-5
$\boldsymbol{x}$	$\boldsymbol{x^3}$		
$\boldsymbol{+3}$			$\boldsymbol{-15}$

   a. Explain how you know that the x^2 and -5 terms can be added to the top of the box as shown?

   b. Use what you have learned about polynomial multiplication using the box method to help Rocky and Mickey find the values missing from each cell as well as the remaining expression across the top.

	x^2		-5
$\boldsymbol{x}$	$\boldsymbol{x^3}$		
$\boldsymbol{+3}$			$\boldsymbol{-15}$

3. From the work in questions 1 and 2, show the complete factorization of $\boldsymbol{x^3+7x^2+7x-15}$.

4. Use the box method that was developed above to divide the following polynomials:

   a. $(x^3 - 2x^2 - 23x + 60) \div (x - 4)$

   b. $(2x^3 + 14x^2 + 19x - 20) \div (x + 4)$

The problems above involved polynomial division that resulted in no remainder. We can use the results of the division to write the polynomial in factored form.

5. Use the results above to express each polynomial in factored form (if possible, factor completely):

   a. $x^3 - 2x^2 - 23x + 60 =$

   b. $2x^3 + 14x^2 + 19x - 20 =$

We have been working with operations on polynomials throughout this module. The results of adding, subtracting, and multiplying were viewed algebraically as well as graphically. In a way, these operations on polynomials are similar to the corresponding operations on integers. When we add or subtract two polynomials, the result is another polynomial. Multiplying two polynomials gives us a new polynomial. The set of all polynomials and the set of all integers are closed under addition, subtraction, and multiplication.

6. Explain the above statement in your own words.

Polynomial division (like division with integers) does not always result in zero as a remainder. When dividing integers, we were able to express the quotient using fractions (example: $5 \div 3 = 1\frac{2}{3}$).

We can do the same with polynomial division. See if you can figure out how this works with the division problem below.

7. $(2x^3 + x^2 - 8x + 5) \div (x + 3)$

Additional Practice Problems

For each problem below, divide using the box method. You may not need all of the cells of the boxes provided. If there is zero remainder, rewrite the polynomial in factored form (factor completely where possible).

1. $(2x^2 - 7x + 10) \div (x - 5)$

2. $(x^3 - 10x^2 + 19x + 30) \div (x - 6)$

3. $(x^3 + 2x^2 - 51x + 108) \div (x + 9)$

4. $(2x^3 - 15x^2 + 34x - 21) \div (x - 1)$

5. $(x^4 - 5x^3 - 8x^2 + 13x - 12) \div (x - 6)$

6. $(x^3 - 5x^2 - 2) \div (x - 4)$

3.7 Warm Up

Bigger Boxes

In the last task, you used polynomial division for linear divisors. The problems below extend your work with nonlinear divisors.

You'll need bigger boxes for these problems. Be sure to look out for any missing terms.

1. $(6x^3 + 13x^2 - 11x - 15) \div (3x^2 - x - 3)$
2. $(18x^4 + 9x^3 + 3x^2) \div (3x^2 + 1)$

3.7 Seeing Structure
A Practice Understanding Task

http://www.flickr.com/photos/nguyendai/479560639

Claire and Carmella were having a discussion about how easy it is to graph polynomial functions. Claire stated: "All you need to know to sketch the graph of a polynomial function is the degree of the polynomial. The degree will tell you the end behavior and the number of times the graph will cross the *x*-axis." Carmella mostly agreed, however, she thought there was something not quite right with this statement.

1. Modify Claire's statement about sketching the graph of a polynomial function:

For each function, identify the end behavior, intercepts, and roots (including the multiplicity) of the function.

2. Equation: $f(x) = -x(x-2)(x-4)$

Roots (including multiplicity):

Intercepts:

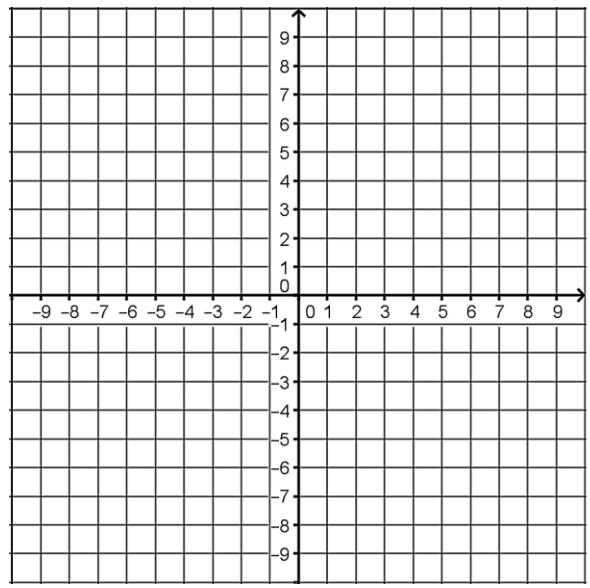
Leading coefficient:

Degree:

End behavior:
As $x \to -\infty,\ f(x) \to$ ______

As $x \to \infty,\ f(x) \to$ ______

Graph:



3. Equation: $f(x) = (x-1)(x^2+4x+4)$

Roots (including multiplicity):

Intercepts:

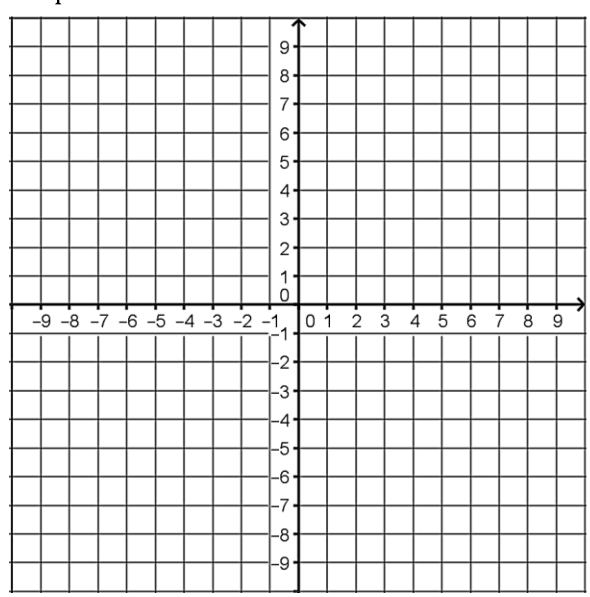
Leading coefficient:

Degree:

End behavior:
As $x \to -\infty,\ f(x) \to$ ______

As $x \to \infty,\ f(x) \to$ ______

Graph:



4. Equation: $f(x) = x^3 - x^2$

Roots (including multiplicity):

Intercepts:

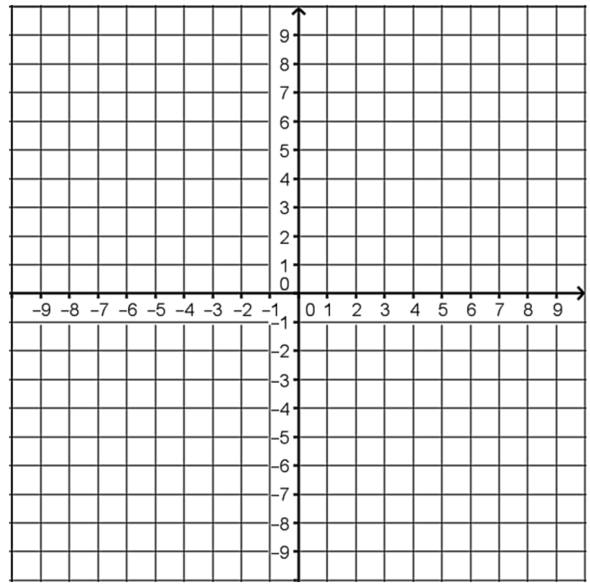
Leading coefficient:

Degree:

End behavior:
As $x \to -\infty$, $f(x) \to$ ______

As $x \to \infty$, $f(x) \to$ ______

Graph:



5. Equation: $f(x) = x^4 - 16$
Hint: $x^4 - 16 = (x^2 + 4)(x^2 - 4)$

Roots (including multiplicity):

Intercepts:

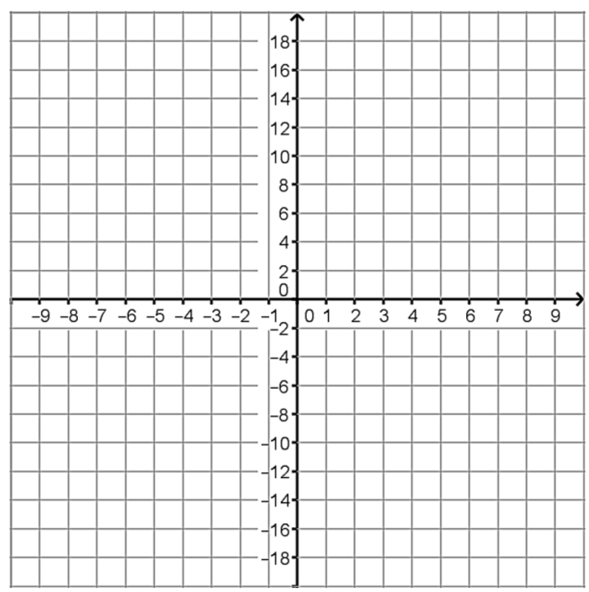
Leading coefficient:

Degree:

End behavior:
As $x \to -\infty$, $f(x) \to$ ______

As $x \to \infty$, $f(x) \to$ ______

Graph:



6. Equation: $f(x) = x^4 + 2x^3$

Roots (including multiplicity):

Intercepts:

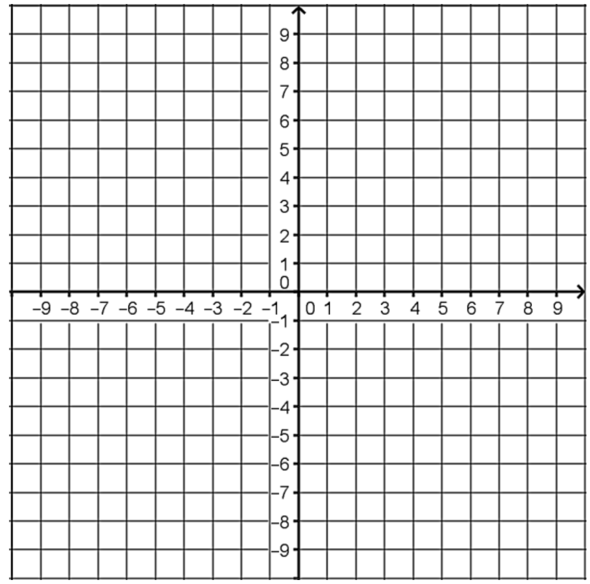
Leading coefficient:

Degree:

End behavior:
As $x \to -\infty$, $f(x) \to$ ______

As $x \to \infty$, $f(x) \to$ ______

Graph:



7. Equation: $f(x) = x^3 - x^2 + 5x - 5$
Hint: $x = 1$ is a root

Roots (including multiplicity):

Intercepts:

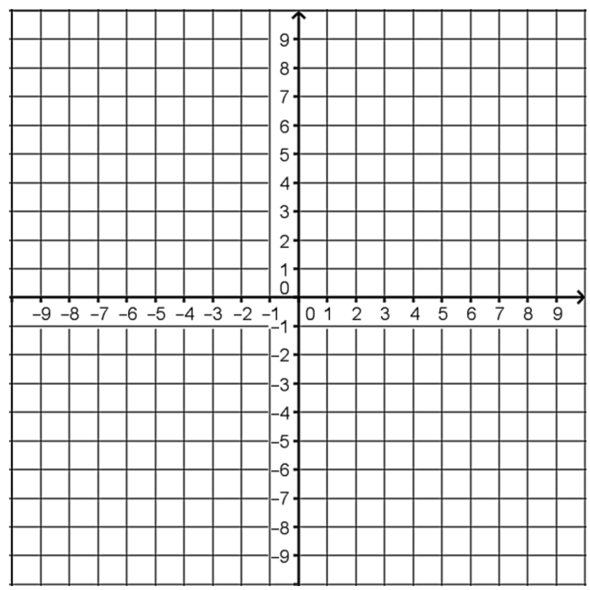
Leading coefficient:

Degree:

End behavior:
As $x \to -\infty$, $f(x) \to$ ______

As $x \to \infty$, $f(x) \to$ ______

Graph:



8. Explain how you are able to graph a polynomial that is not already in factored form?

9. If you know one root of a cubic function, how do you find the other roots? Explain.

10. Return to question 1. Make any additions/alterations to your statement(s).

3.8 Warm Up
Boxing Match

Use the box method to perform the following operations (you will need to draw your own boxes):

1. $(6x^3 + 13x^2 - 4x - 15) \div (2x + 3)$

2. $(x^3 - 7x^2 + 3x) \cdot (3x + 2)$

3. $(x - 5)(x^2 - 7x + 1)$

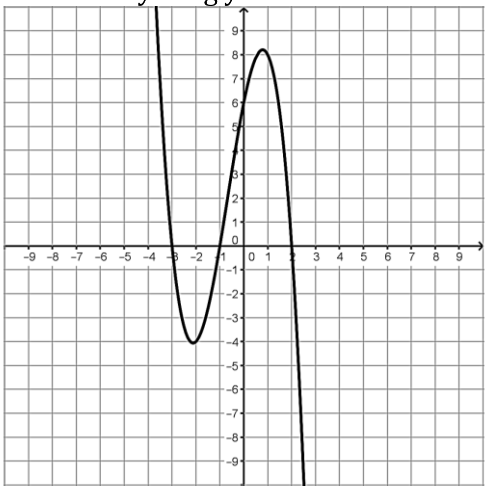
4. $(3x^3 + 20x^2 - 2x + 35) \div (x + 7)$

3.8 Graphing All Poly's
A Solidify Understanding Task

http://www.flickr.com/photos/cu2nite/3477888658

Part I: Connecting the number system to polynomials.

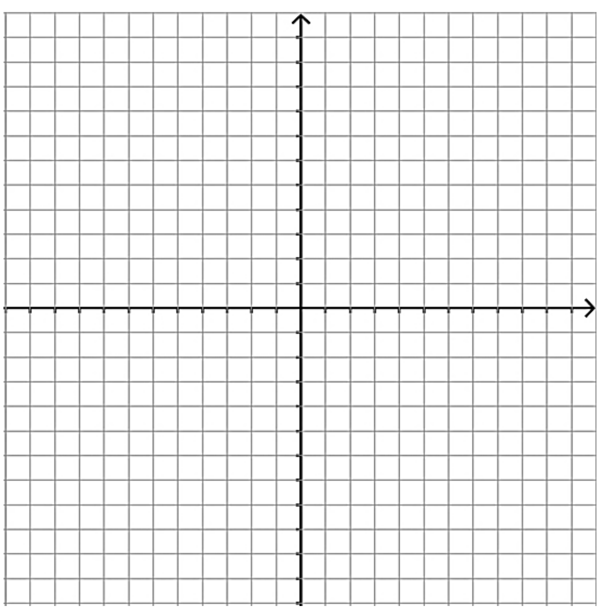
1. Write everything you know about the following polynomial:



2. In case this was not part of what you wrote in question 1, use function notation to highlight values of importance for this function. (For example: $f(0) = 6$)

3. How can we use the original function and one of its factors to find the remaining factors and sketch a graph? Use the problem below to explain your method.

$f(x) = x^3 + 4x^2 + x - 6$ has a factor of $(x - 1)$ <u>Explanation:</u>



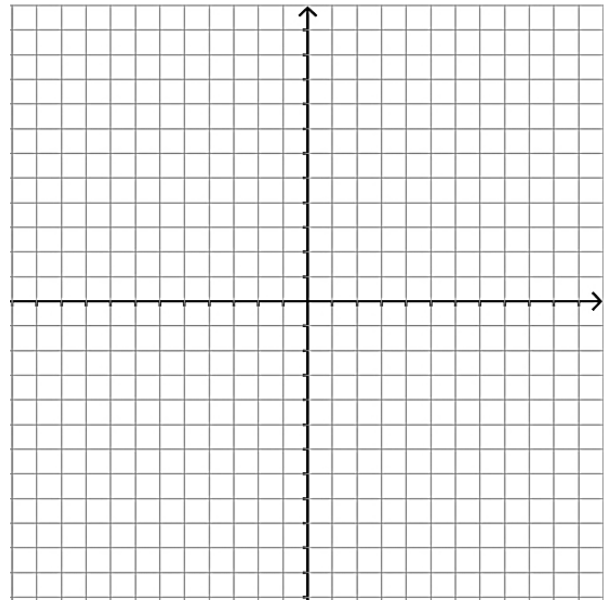
Part II

For each cubic function, one factor is given. Do your best to find the remaining factors, use this information to determine all roots of the function, and sketch a graph.

4. Function: $f(x) = x^3 + 3x^2 - 4x - 12$
   Factor: $(x + 3)$

   Roots of function:

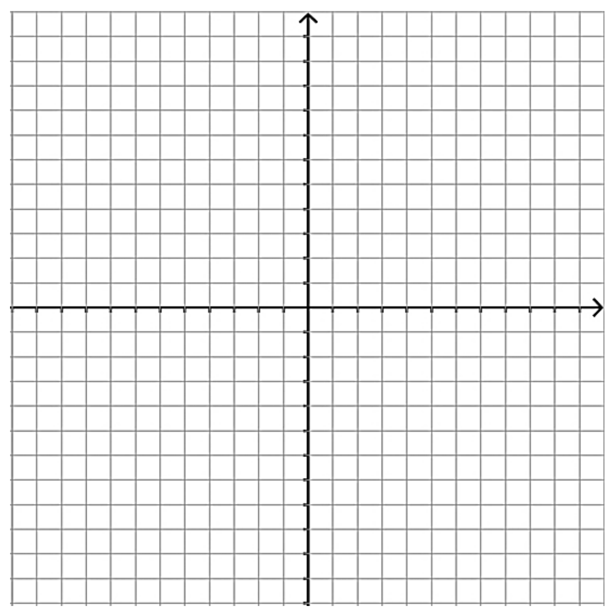
Graph:



5. Function: $f(x) = x^3 + 5x^2 + 8x + 4$
   Factor: $(x + 1)$

   Roots of function:

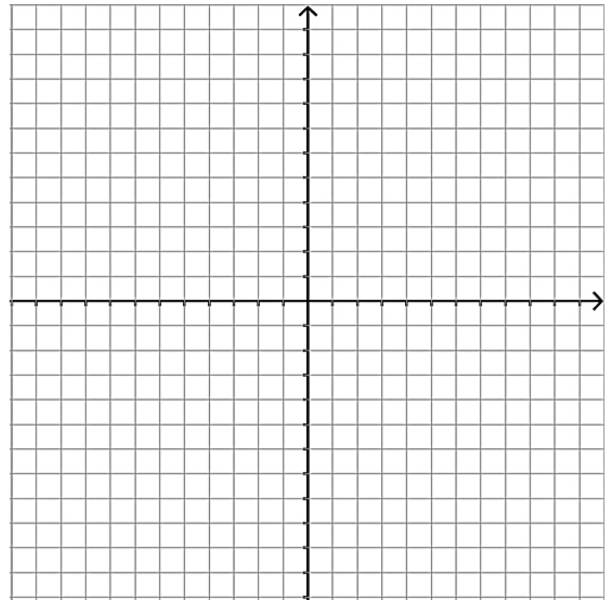
Graph:



6. Function: $f(x) = 2x^3 - 7x^2 + 2x + 3$
   Factor: $(x - 3)$

   Roots of function:

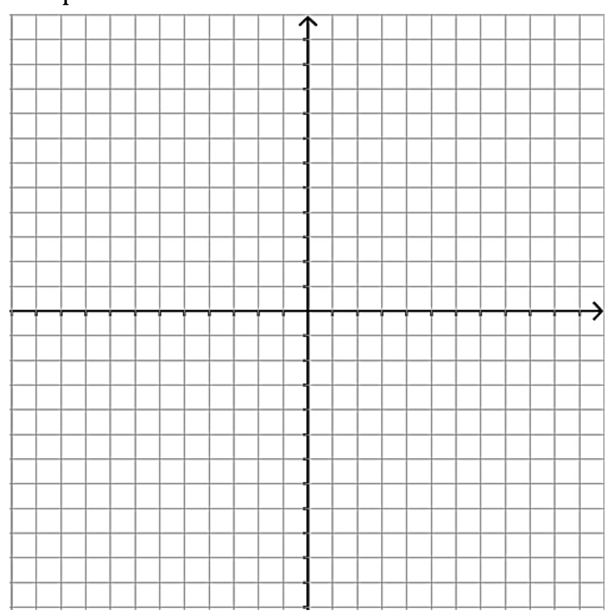
Graph:



7. Function: $f(x) = x^3 - x^2 + 4x - 4$
   Factor: $(x^2 + 4)$

   Roots of function:

Graph:

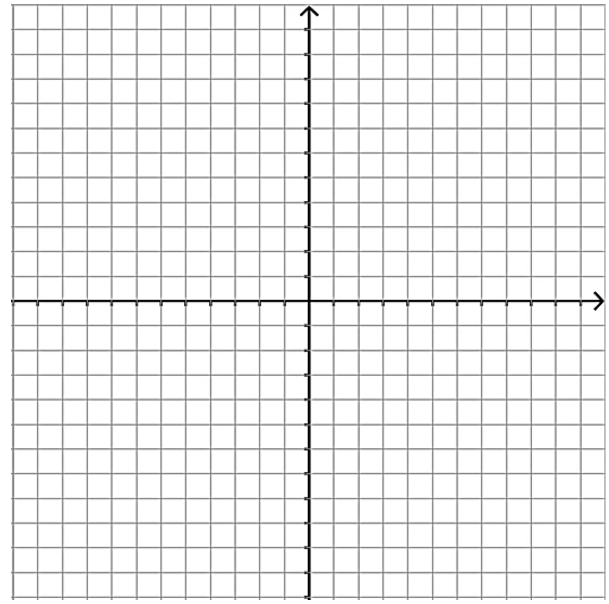


8. Function: $f(x) = x^3 - 6x^2 + 13x - 10$
$f(2) = 0$

What does $f(2) = 0$ mean? How can this help determine the factors?

Roots of function:

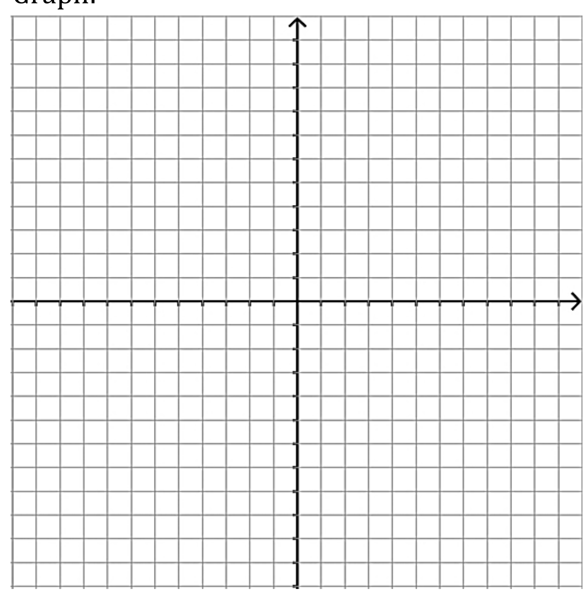
Graph:



9. Function: $f(x) = x^4 + 9x^3 + 17x^2 - 9x - 18$
$f(-3) = 0$

Roots of function:

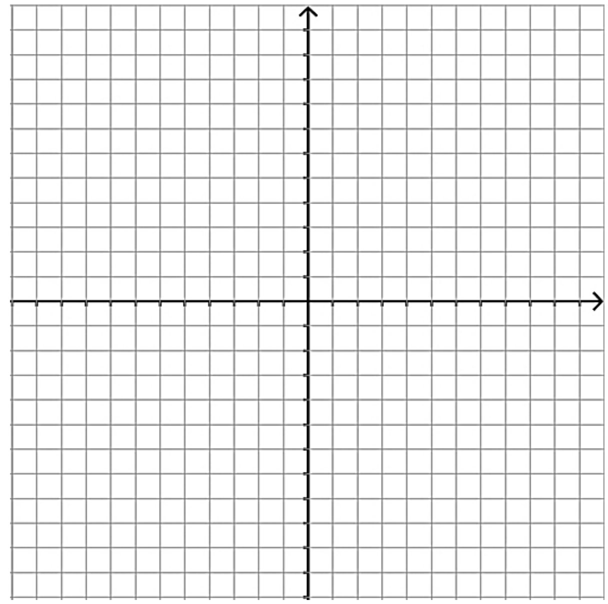
Graph:



10. Find all linear factors and graph: $f(x) = x^4 - 1$

Roots of function:

Graph:



11. If you know a linear factor or a root of the polynomial function, how can you find the remaining factors and roots?

Remainder Theorem: If $\boldsymbol{p(r) = 0}$, then $\boldsymbol{(x - r)}$ is a factor of the polynomial $\boldsymbol{p(x)}$.

12. How does the Remainder Theorem help in finding roots of a polynomial function?

3.9 Warm Up
It's All in Your Imagination

Use the given information to factor the polynomial completely. Then, use the factorization to find all the zeros/roots of the function.

1. $f(x) = x^3 + x^2 + 4x + 4$
   $(x + 1)$ is a factor
   $y = 4$ is the y-intercept
   $(1, 10)$ is on the graph

2. $g(x) = 4x^3 - 12x^2 + x - 3$
   $(1, -10)$ is on the graph
   $y = -3$ is the y-intercept
   $g(3) = 0$

3.9 Calling All Poly's
A Solidify Understanding Task

http://www.flickr.com/photos/cu2nite/3477888658

Definition: The **conjugate** is where you change the sign in the middle of two terms like this:

$3 + x$ and $3 - x$ are conjugates. $4 + \sqrt{7}$ and $4 - \sqrt{7}$ are conjugates.

Given the roots, find the factors and write a polynomial equation of least degree in standard form.

1. Roots: $3, -4,$ and 0

2. Roots: $5, 2i, -2i$

3. Roots: $\sqrt{3}, -\sqrt{3}, -2$

4. Roots: $2, 3i$ and ______.

5. Roots: $-6, 3 + 2i$, and ______.

6. In the questions above, are there other polynomial equations that would produce the same roots? Explain.

Find the polynomial function with least degree and rational coefficients that satisfies the given conditions.

7. $f(2) = 0, f(-5i) = 0, f(0) = 750$

8. Roots: $\sqrt{7}, 3$ and passes through the point $(2, 6)$

9. $f(-2) = 0, f(i) = 0, f(1) = 0, f(2) = 12$

10. What have you learned about polynomial functions as a result of this task? Discuss concepts such as roots (rational, irrational, imaginary), intercepts, end behavior, factors of the polynomial function.

3.10 I Know, What Do You Know?
A Practice Understanding Task

http://www.flickr.com/photos/chrisbrenschmidt/18319558

Use the information provided to graph and write out the polynomial function in factored form. Note: you may have to determine other roots based upon what is given.

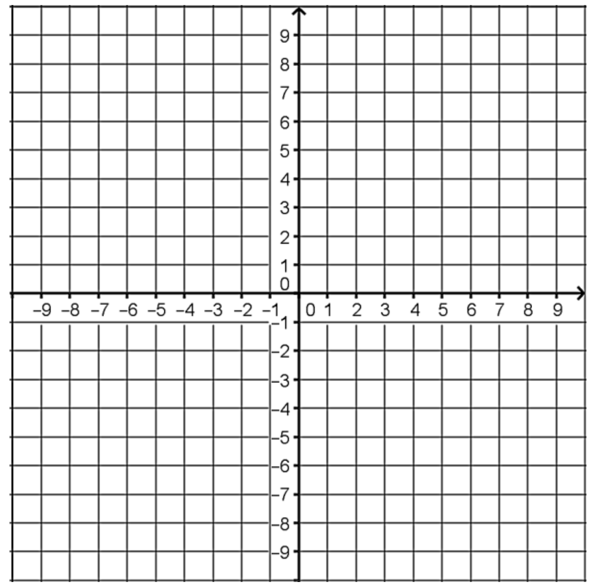
1. Degree of the polynomial: 3
   Given roots: $-2, 1, 1$
   Leading coefficient: -2

   Additional roots:

   Equation factored form:

   Equation in standard form:

   Graph:



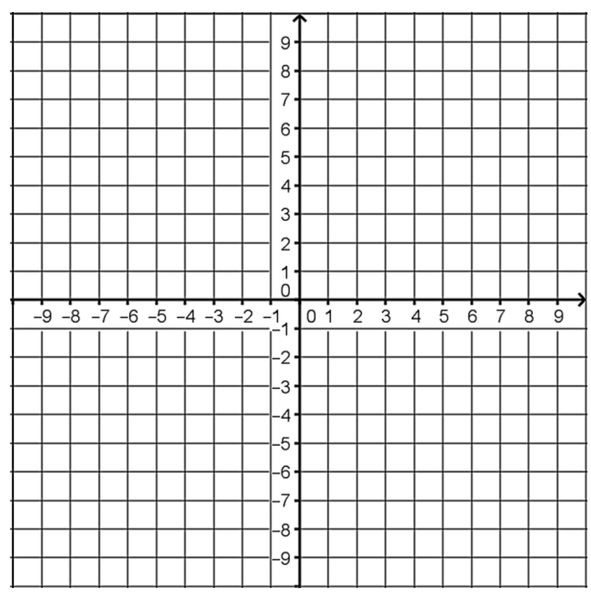
2. Degree of the polynomial: 4
   Given roots: $2 + i, 4, 0$
   Leading coefficient: 1

   Additional roots:

   Equation factored form:

   Equation in standard form:

   Graph:



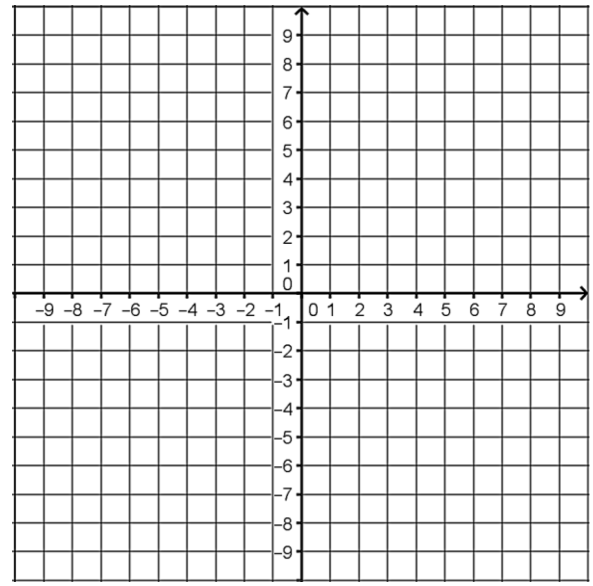
3. Degree of the polynomial: 2
Given roots: $\sqrt{2}$
Leading coefficient: -1

Additional roots:

Equation factored form:

Equation in standard form:

Graph:



If I know…What do you know? For each problem, what I know about a function is given. Your job is to complete the table of information with what you know.

4. **Function:**
$f(x) = 2(x-1)(x+3)^2$

End Behavior:
As $x \to -\infty$, $f(x) \to$ __

As $x \to \infty$, $f(x) \to$ ____

Roots (with multiplicity):

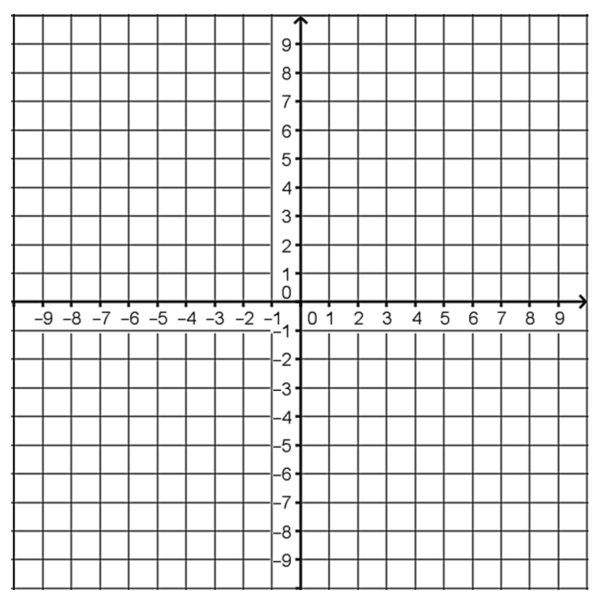
Value of Leading Coefficient:

Degree:

Domain:

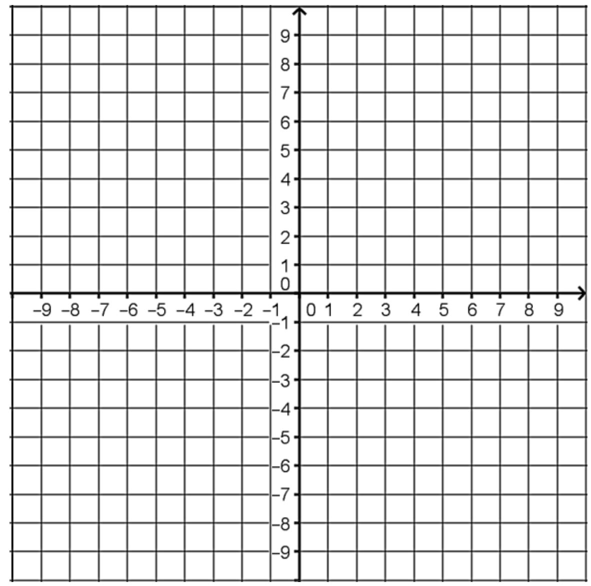
Range: All real numbers

Graph:



5. **Function:**

Graph:



End Behavior:
As $x \to -\infty$, $f(x) \to$ __∞__

As $x \to \infty$, $f(x) \to$ ____

Roots (with multiplicity):
$(3, 0), m = 1$
$(-1, 0), m = 2$
$(0, 0), m = 2$

Value of Leading Coefficient: -1

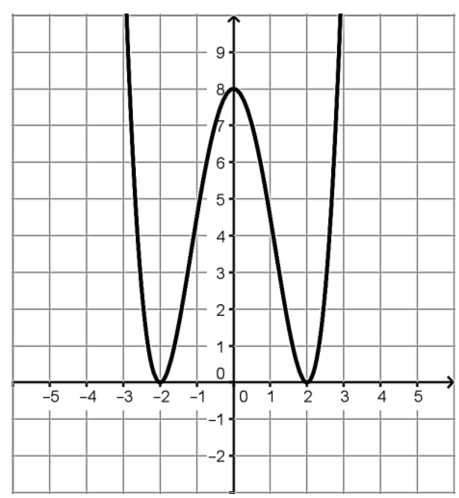
Degree:

Domain:

Range:

6. **Function (hint: the leading coefficient $\neq$ 1):**

Graph:



End Behavior:
As $x \to -\infty$, $f(x) \to$ __

As $x \to \infty$, $f(x) \to$ ____

Roots (with multiplicity):

Value of Leading Coefficient:

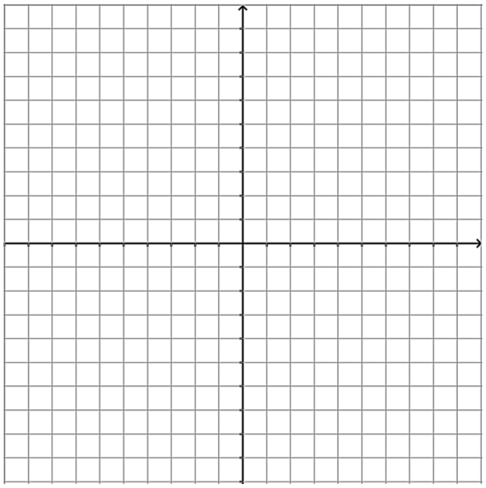
Degree:

Domain:

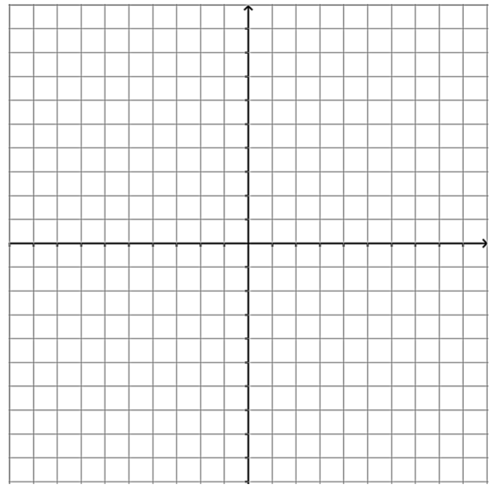
Range:

Without using technology, sketch the graph of the polynomial function described. The term "imaginary roots" means complex zeros.

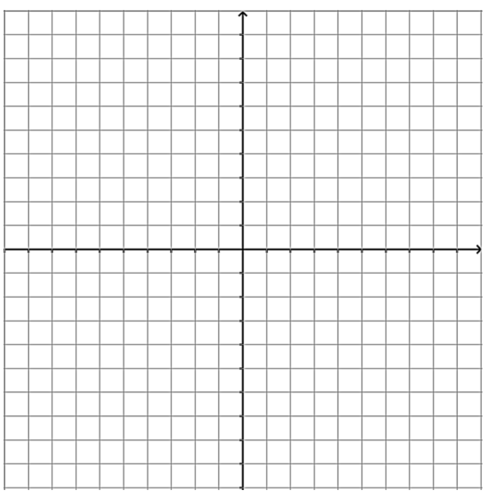
7. A cubic function with a leading coefficient of -2, with two negative zeros and one positive.



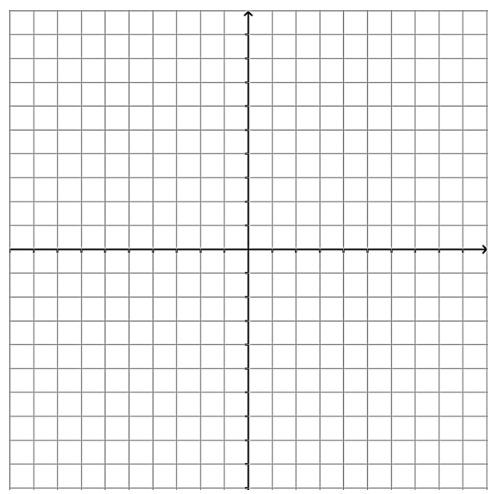
8. A cubic function passing through the point $(0, 4)$, with one negative zero and one positive double zero.



9. A quartic function with a leading coefficient of -3, with two imaginary roots and one positive double root.



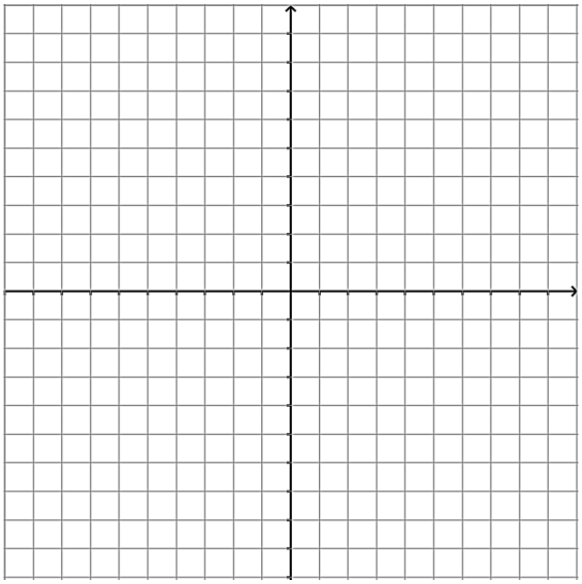
10. A quartic function with a leading coefficient of 2, with two negative zeros and one positive double root.



Find all factors and sketch the graph of the polynomial functions.

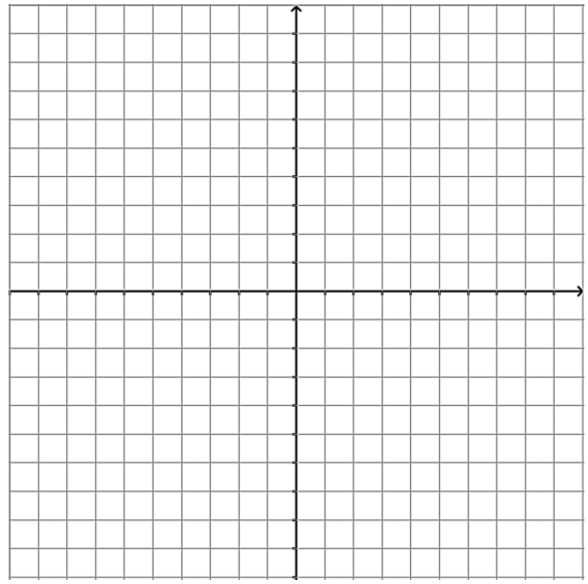
11. $f(x) = x^3 - x^2$

Factors:



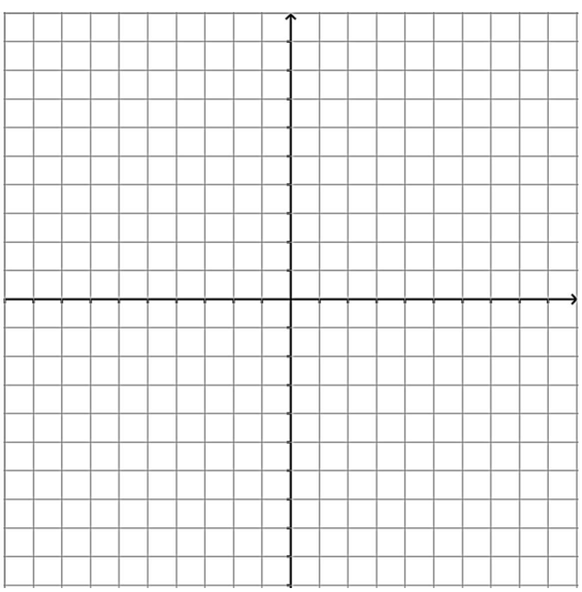
12. $f(x) = x^4 - x^2$

Factors:



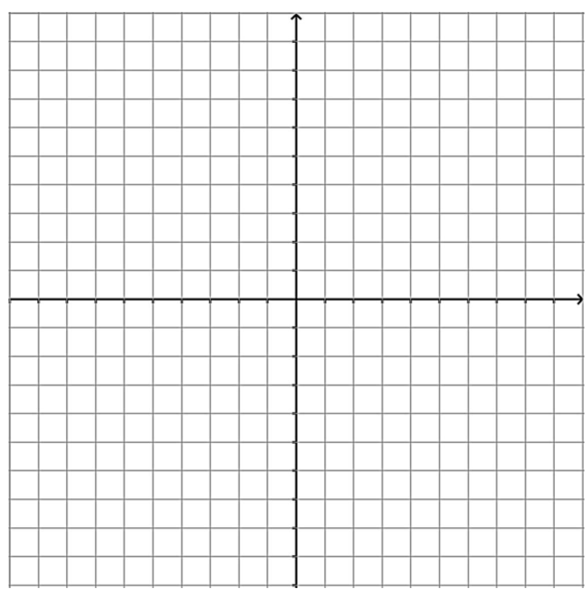
13. $f(x) = x^3 - 2x$

Factors:



14. $f(x) = x^3 - x^2 + 9x - 9$

Factors:



Integrated Math 3
Module 3
Polynomial Functions
Ready, Set, Go! Homework

Adapted from

The Mathematics Vision Project:
Scott Hendrickson, Joleigh Honey, Barbara Kuehl,
Travis Lemon, Janet Sutorius

Ready, Set, Go!

Ready

Topic: Inequality statements.

Which is greater? For each problem, make a true statement by placing the appropriate inequality symbol between the two expressions.

If $a > b$, then:

1. $3a$____$3b$
2. $b - a$___$a - b$
3. $a + x$___$b + x$

If $x > 10$, then:

4. x^2___2^x
5. $\sqrt{x}$___x^2
6. x^2___x^3

Set

Topic: Types of functions

Determine the type of function for each problem. Explain how you know.

7.

x	$f(x)$
1	3
2	6
3	9
4	12
5	15

8.

x	$f(x)$
1	3
2	6
3	12
4	24
5	48

9.

x	$f(x)$
1	3
2	9
3	18
4	30
5	45

10.

x	$f(x)$
1	3
2	12
3	30
4	60
5	105

11. $f(x) = -2x^3 + 3x^2 - 5$

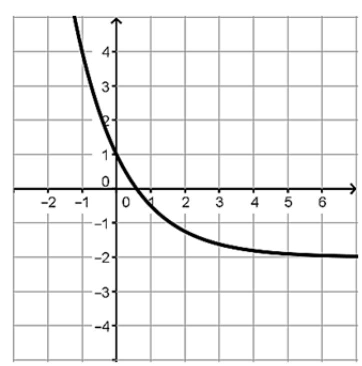
12. $f(x) = x^2 - 9$

13. $g(x) = \log_2(x + 3)$

14. $g(x) = 2(x - 4) + 7$

15. $h(x) = 2 \cdot 3^x + 1$

16.



Go

Topic: Combining functions.

Use the given functions to solve problems 16 – 22.

$\boldsymbol{f(x) = x - 3}$ $\boldsymbol{g(x) = x + 2}$ $\boldsymbol{h(x) = -x + 1}$

$\boldsymbol{m(x) = x^2 + 3x + 2}$ $\boldsymbol{n(x) = 2x^3 - x^2 + 2x + 1}$ $\boldsymbol{p(x) = 2x + 1}$

17. $f(x) + g(x)$

18. $f(x) - h(x)$

19. $f(x) + p(x)$

20. $g(x) + h(x)$

21. $m(x) + g(x)$

22. $n(x) + m(x)$

23. $m(x) - g(x)$

24. $h(x) - m(x)$

Determine if the following statements are ALWAYS or NEVER true. If the statement is NOT true, rewrite it so that it is ALWAYS TRUE.

25. The sum of two linear functions is another linear function.

26. The sum of a linear and a quadratic is a cubic function.

27. The sum of a cubic and a quadratic function is a cubic function.

28. The sum of two functions is always a function (passes the vertical line test).

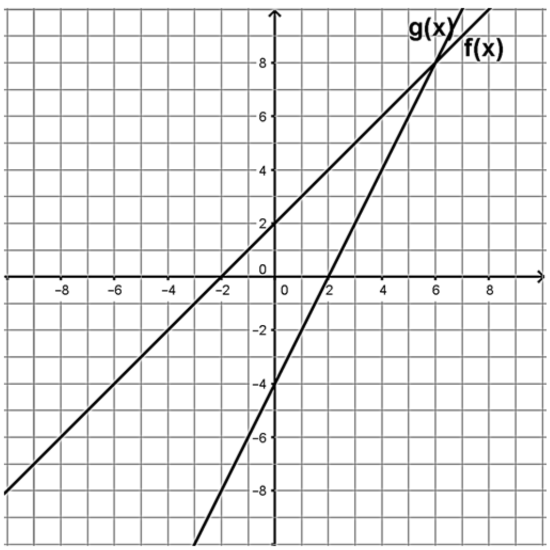
29. The product of two functions is always a function (passes the vertical line test).

Ready, Set, Go!

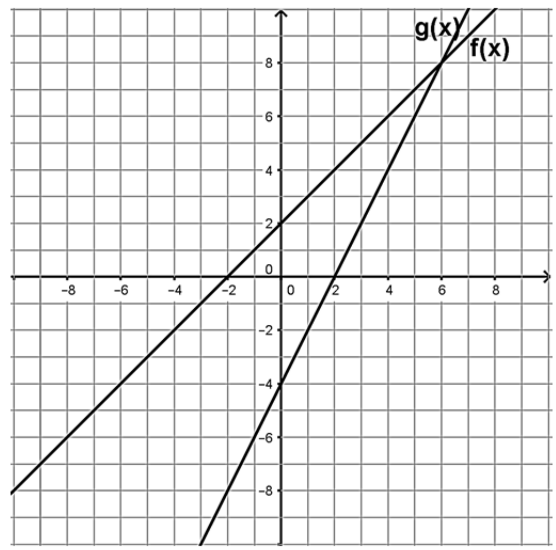
Ready
Topic: Combining polynomial functions graphically.

Use the graphs of $f(x)$ and $g(x)$ to sketch the graph of the following:

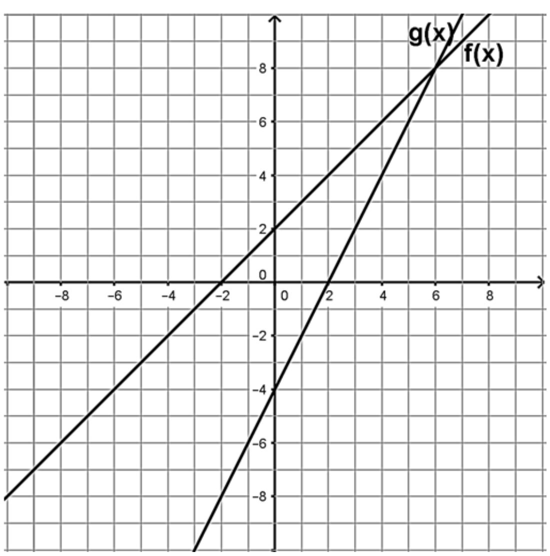
1. $f(x) + g(x)$



2. $f(x) - g(x)$



3. $f(x) \cdot g(x)$



4. Complete each sentence below.
   a. The sum of two linear functions is…
   b. The difference of two linear functions is…
   c. The product of two linear functions is…

Set

Topic: Ordering real number expressions

Order the following numbers from least to greatest without using a calculator.

5. 100^3	$\sqrt{100}$	$\log_2 100$	100
6. 2^{-1}	$\sqrt{100}$	$\log_2\left(\frac{1}{8}\right)$	0
7. 2^0	$\sqrt{16}$	$\log_2 8$	2

Which is greater? For each problem, make a true statement by placing the appropriate inequality symbol between the two expressions. (Hint: think about what you know about the expression and the end behavior as well as rates of change of a function instead of plugging in values).

If $x < -100$, *then:*

8. x^2____2^x

9. x^5___x^2

10. x^2___x^3

If $x > 100$, *then:*

11. x^2___2^x

12. x^5____x^2

13. x^2___x^3

Go

Topic: Combining functions

Perform each operation. Write your answers in standard form.

14. $f(x) = x^5 + 3x^2 + 4x^4, g(x) = 3x^5 - x^3 + 3x^2$
    $f(x) + g(x) =$

15. $f(x) = x^3 - 4x^2, g(x) = 4x^3 + 3x^2 - x + 6$
    $f(x) - g(x) =$

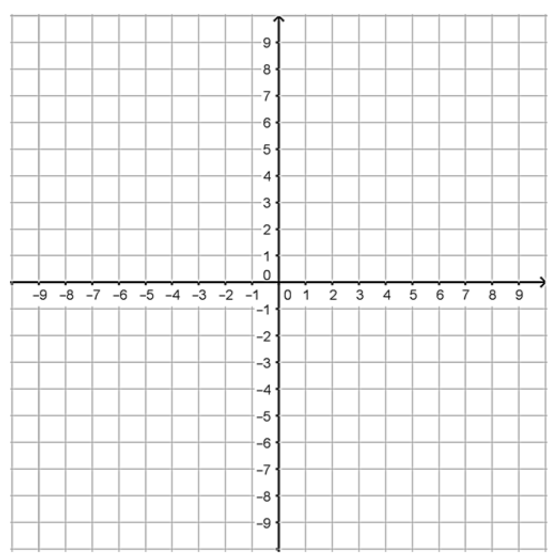
16. $f(x) = 3x^2 + 4x, g(x) = x^2 - 5$
    $f(x) \cdot g(x) =$

17. $f(x) = x^4 - 6x^2 + 5x^3, g(x) = 2x^2 - 7x^4 + 6$
    $g(x) - f(x) =$

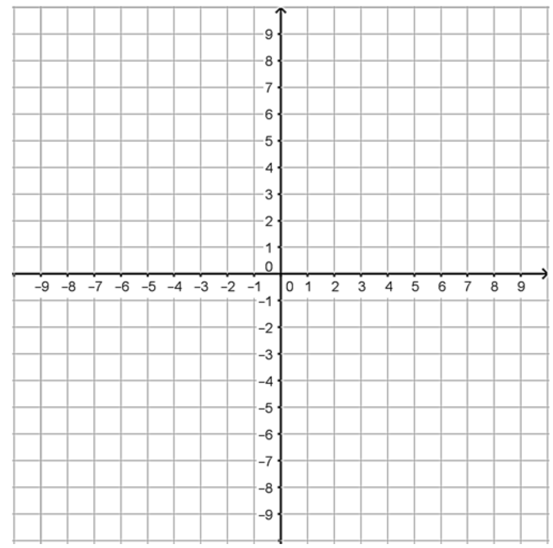
Graph each set of functions on the same axes. Label each function and state how the functions are related to the graphs of their parent functions.

18. $f(x) = 2^x + 2$
    $g(x) = x + 2$
    $h(x) = x^2 + 2$

    Features in common:

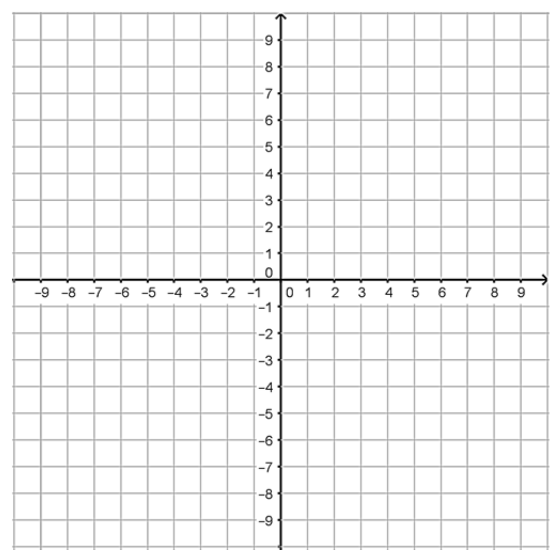


19. $f(x) = 3\sqrt{x-2}$
$g(x) = 3(x-2)$
$h(x) = 3(x-2)^2$



Features in common:

20. $f(x) = \frac{1}{2}|x-1| - 2$
$g(x) = \frac{1}{2}(x-1) - 2$
$h(x) = \frac{1}{2}(x-1)^2 - 2$



Features in common:

Ready, Set, Go!

Ready

Topic: Forms of linear and quadratic functions

The different forms of linear and quadratic functions are listed below. Determine what features of the function/graph can quickly be determined based upon the structure of each form of linear and quadratic functions.

Linear

1. Standard form: $ax + by = c$

2. Slope-intercept form: $y = mx + b$

3. Point-slope form: $y = m(x - x_1) + y_1$

Quadratic

4. Standard form: $y = ax^2 + bx + c$

5. Factored form: $y = a(x - r_1)(x - r_2)$

6. Vertex form: $y = a(x - h)^2 + k$

For each, write what you know about the function (including end behavior) and then graph.

7. Equation: $f(x) = (x - 2)(x + 3)$
What I know about this function:

Intercepts:

Domain:

Range:

Maximum(s):

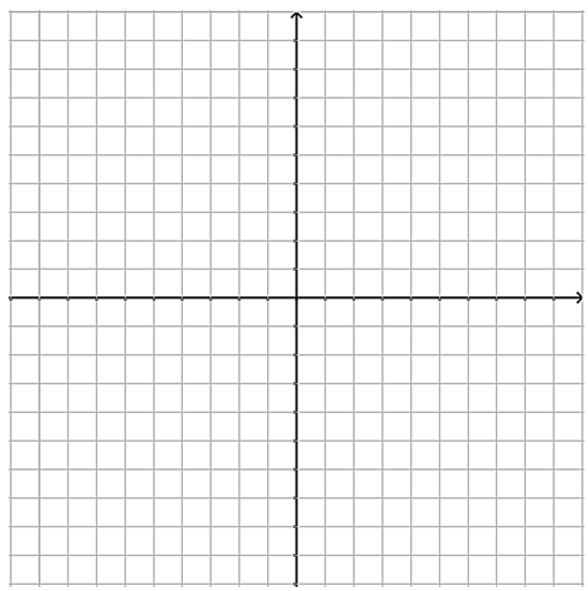
Minimum(s):

Intervals of increase/decrease:

End behavior:
As $x \to -\infty$, $f(x) \to$ ______

As $x \to \infty$, $f(x) \to$ ______.

Graph:



8. Equation: $g(x) = x^2 + 4x + 4$
What I know about this function:

Intercepts:

Domain:

Range:

Maximum(s):

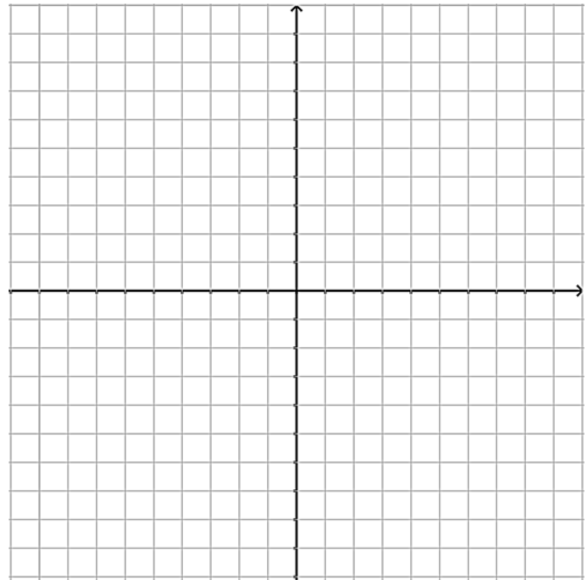
Minimum(s):

Intervals of increase/decrease:

End behavior:
As $x \to -\infty$, $g(x) \to$ ______

As $x \to \infty$, $g(x) \to$ ______

Graph:



9. Equation: $y = -x^2 - 1$
What I know about this function:

Intercepts:

Domain:

Range:

Maximum(s):

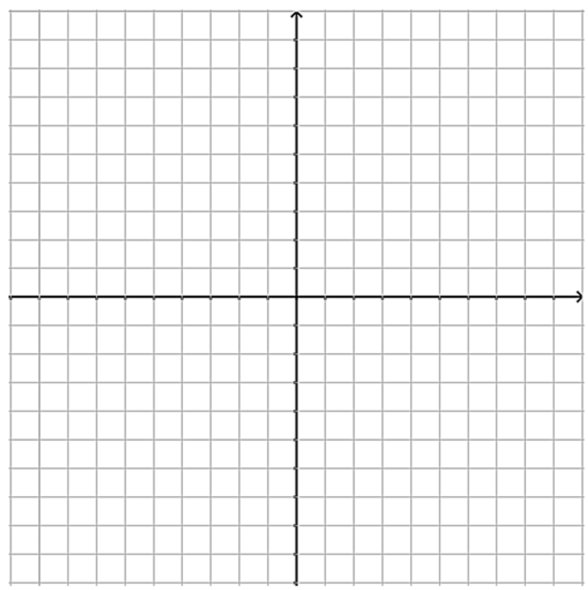
Minimum(s):

Intervals of increase/decrease:

End behavior:
As $x \to -\infty$, $y \to$ ______

As $x \to \infty$, $y \to$ ______

Graph:



<table>
<tr>
<td>10. Equation: $h(x) = 2(x-3)^2 + 1$
What I know about this function:

Intercepts:

Domain:

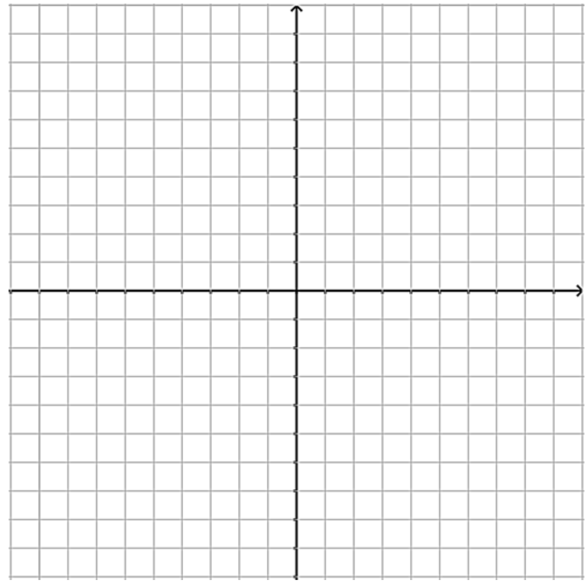
Range:

Maximum(s):

Minimum(s):

Intervals of increase/decrease:

End behavior:
As $x \to -\infty,\ h(x) \to$ ______

As $x \to \infty,\ h(x) \to$ ______</td>
<td>Graph:
</td>
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</table>

Set

Topic: End behavior of various types of functions

Determine the function type and state the end behavior in the form as $x \to$ __, $f(x) \to$ __.

11. $f(x) = x^2 + 12x - 1$

12. $g(x) = 4 \cdot 2^x$

13. $h(x) = -x^3 + 1$

14. $p(x) = -x^2 + 3x - 1$

Use the equations in questions 11-14 to answer the following:

15. Which function above has the greatest value at $x = 1{,}000$?

16. Which function above is *always* increasing?

17. Which function above is *always* decreasing?

18. Which function above has a maximum value?

19. Which function above has a minimum value?

Go

Topic: Solving polynomial equations

Solve for *x*.

20. $x^2 - 16 = 0$

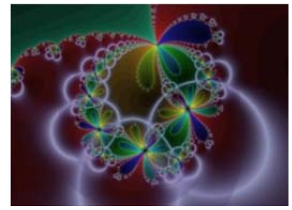
21. $2x^2 + 4x + 3 = 0$

22. $3x^2 - 5x - 6 = 0$

23. $3x^2 - 4x = 12$

24. $(x + 4)(x - 3)(x + 1) = 0$

25. $x(x^2 - 6x + 9) = 0$



Ready, Set, Go!

Ready

Topic: Solving polynomial equations using factors.

Solve for x.

1. $x^2 - 25 = 0$
2. $x^2 + 8x + 7 = 0$
3. $8x^2 - 14x - 9 = 0$
4. $3x^2 + 5x = 2$
5. $(2x - 1)(x + 4)(5x + 2)=0$
6. $4x(4x^2 + 4x + 1) = 0$

Set

Topic: Combining polynomial functions.

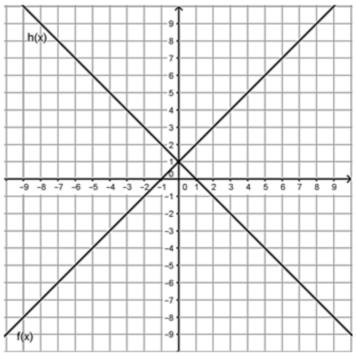
Given $f(x) = x^2 + 3x + 2$ and $g(x) = 5x - 4$, find:

7. $f(x) + g(x)$
8. $f(x) - g(x)$
9. $f(x) \cdot g(x)$

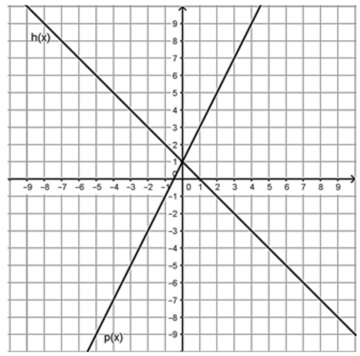
Graphs of the individual functions are given. Graph the solution on the same set of axes.

$\boldsymbol{f(x) = x + 1}$ $\boldsymbol{h(x) = -x + 1}$ $\boldsymbol{p(x) = 2x + 1}$ $\boldsymbol{m(x) = x^2}$

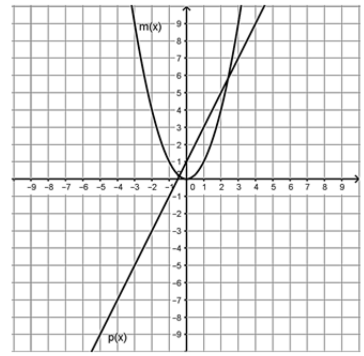
10. $f(x) + h(x)$



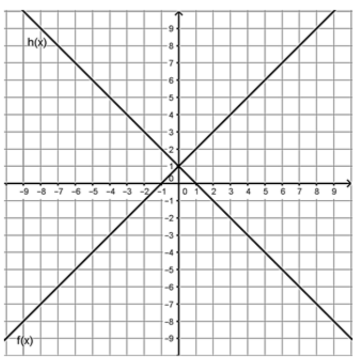
11. $h(x) + p(x)$



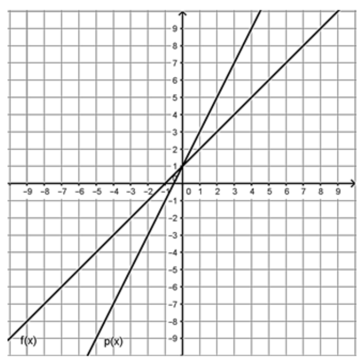
12. $m(x) + p(x)$



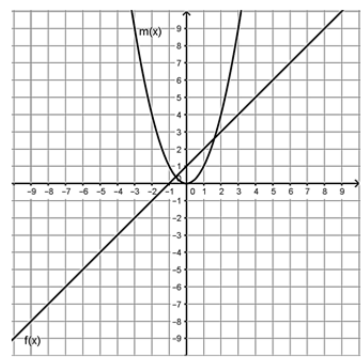
13. $f(x) \cdot h(x)$



14. $f(x) \cdot p(x)$



15. $f(x) \cdot m(x)$



Go

Topic: Simplifying expressions containing exponents

16. $\frac{6x^3y^5}{9yz^4}$

17. $\left(\frac{2x^{-2}y^3z^4}{yz^4}\right)^2$

18. $\frac{6x+9}{3}$

Topic: Solving logarithmic and exponential equations.

Solve each equation.

19. $\log_2(3x - 5) = \log_2(x + 17)$

20. $64^{x-1} = 512$

21. $\log_3(6x + 9) = 5$

22. $\left(\frac{1}{3}\right)^x = 81^{2x-3}$

Topic: Multiplying polynomials.

Multiply each. Simplify solutions by combining like terms

23. $(a + b)(a + b)$

24. $(x - 3)(x^2 + 3x + 9)$

25. $(x - 5)(x^2 + 5x + 25)$

26. $(x + 1)(x^2 - x + 1)$

27. $(x + 7)(x^2 - 7x + 49)$

28. $(a - b)(a^2 + ab + b^2)$

29. Using the patterns from questions 23-28, what do you think you are the factors of $x^3 - 1$? Check your factorization by multiplying the factors together.

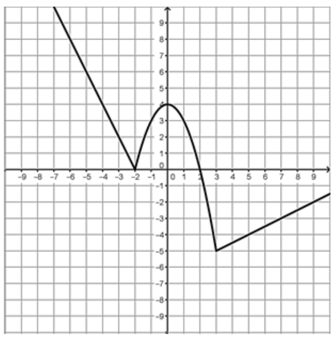
Ready, Set, Go!

Ready

Topic: Describe the features of various functions.

Identify the features of the following functions. (Features include domain, range, intercepts, and end behavior).

1.



Domain:

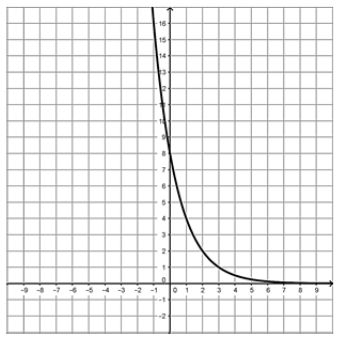
Range:

x-intercepts:

y-intercept:

End Behavior:

2.



Domain:

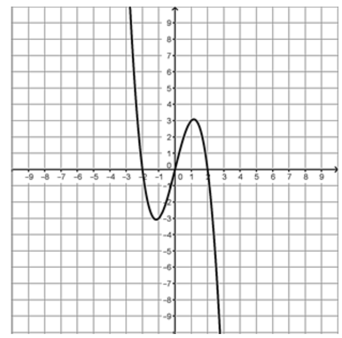
Range:

x-intercepts:

y-intercept:

End Behavior:

3.



Domain:

Range:

x-intercepts:

y-intercept:

End Behavior:

Set

Topic: Features of polynomial functions

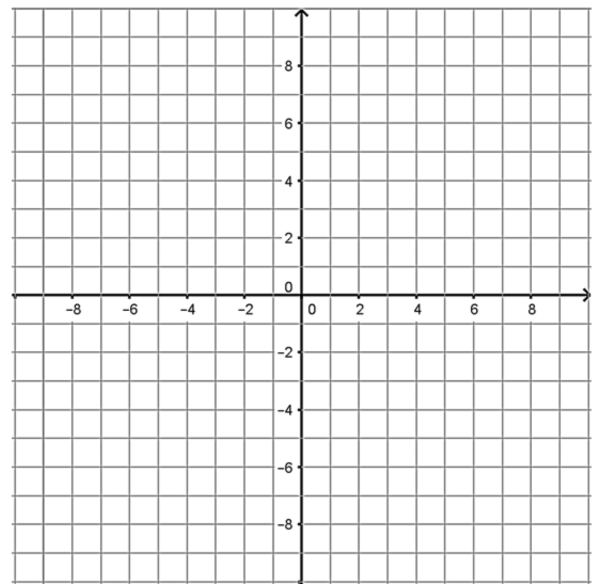
Write the key features of each function (intercepts, end behavior, and where the function is increasing/decreasing), then graph.

4. Equation: $f(x) = (x - 1)^2$
   What I know about this function:

   End behavior:
   As $x \to -\infty,\ f(x) \to$ ______

   As $x \to \infty,\ f(x) \to$ ______

Graph:

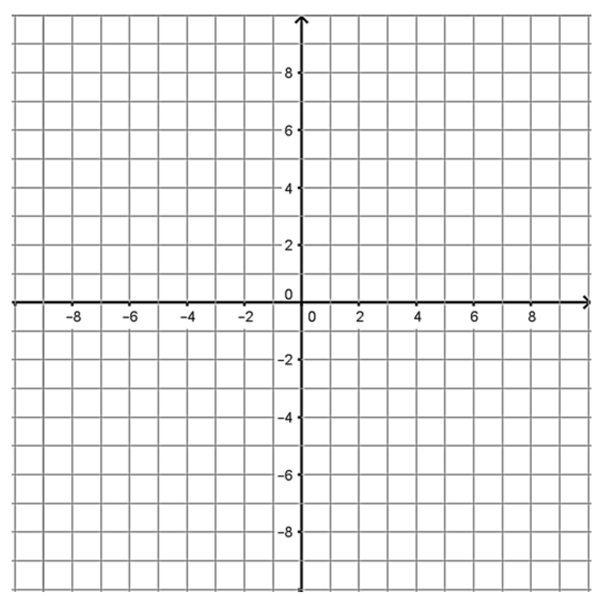


5. Equation: $h(x) = -x^2 + 1$
   What I know about this function:

   End behavior:
   As $x \to -\infty,\ h(x) \to$ ______

   As $x \to \infty,\ h(x) \to$ ______

Graph:

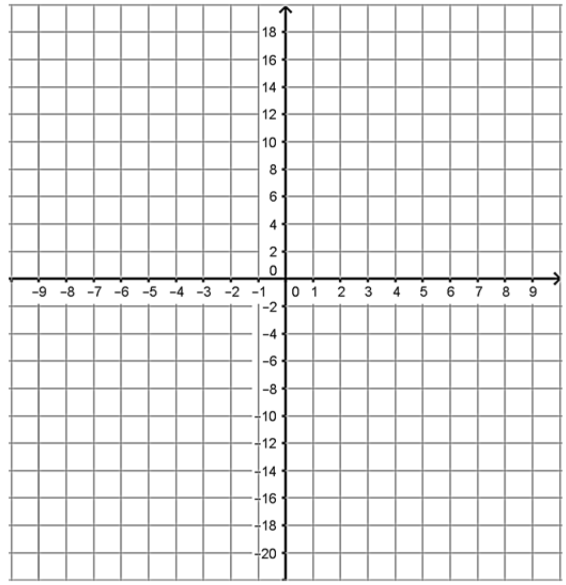


6. Equation: $h(x) = (x-3)(x+4)(x+1)$
What I know about this function:

End behavior:
As $x \to -\infty,\ h(x) \to$ ______

As $x \to \infty,\ h(x) \to$ ______

Graph:

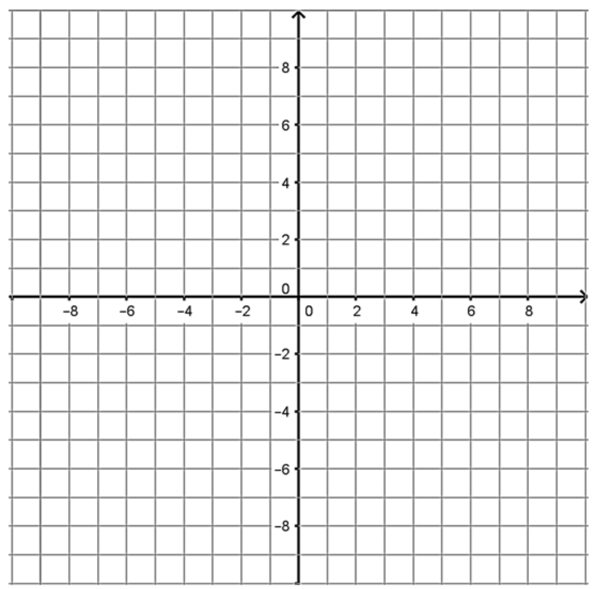


7. Equation: $f(x) = x^3$
What I know about this function:

End behavior:
As $x \to -\infty,\ f(x) \to$ ______

As $x \to \infty,\ f(x) \to$ ______

Graph:



Go

Topic: Comparing functions in different forms.

Use functions a-h to answer the questions below.

a. $f(x) = 3 - 2x$

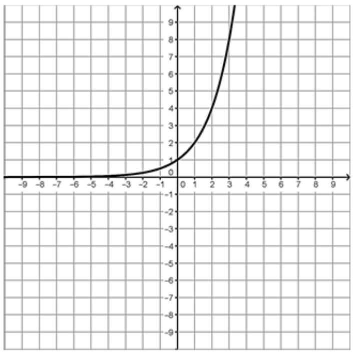
b. $f(x) = \log_2 x$

c. $f(x) = \sqrt{x+1}$

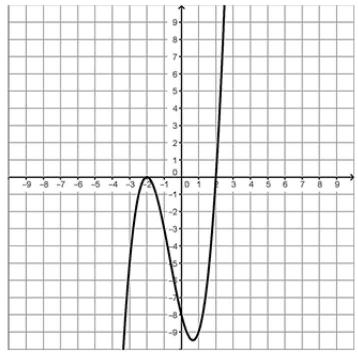
d. $f(x) = 3(x-1)(x+2)(x-4)$

e. $f(x) = -2x^3 + 2x^2 - x + 5$

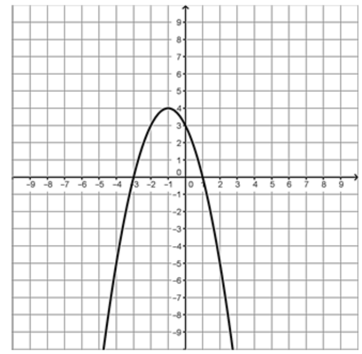
f.



g.



h.



8. Which function(s) do not have a domain of all real numbers?

9. Which function(s) do not have a range of all real numbers?

10. Which function(s) have exactly two *x*-intercepts?

11. Compare *a* and *c*: which has the greatest value as $x \to \infty$?

12. Compare *d* and *f*: which has the greatest value as $x \to \infty$?

13. Compare *f* and *g*: which has the greatest value as $x \to \infty$?

14. Compare *e* and *h*: which has the greatest value as $x \to \infty$?

15. Compare *g* and *h*: which has the highest relative maximum value?

16. Compare *b* and *f*: which has the greatest average rate of change over the interval $[15, 20]$?

Ready, Set, Go!

Ready

Topic: Introduction to the Remainder Theorem

Perform the indicated division using the boxes provided. Be sure to write solutions that contain remainders using fractions.

1. $(2x^3 - 3x^2 - 11x + 6) \div (x - 3)$

2. $(x^3 + 4x^2 - 5x + 5) \div (x - 3)$

3. $(x^3 + 7x^2 + 11x - 3) \div (x + 3)$

4. $(5x^3 + 6x + 8) \div (x + 2)$

Evaluate the following polynomials.

5. $f(x) = 2x^3 - 3x^2 - 11x + 6, f(3) =$

6. $f(x) = x^3 + 4x^2 - 5x + 5, f(3) =$

7. $f(x) = x^3 + 7x^2 + 11x - 3, f(-3) =$

8. $f(x) = 5x^3 + 6x + 8, f(-2) =$

9. a. What do you notice about the solutions from questions 5-9 and the remainders from questions 1-4?

b. Complete the statement below:

"If $f(a) = k$, then $f(x) \div (x - a)$ will have remainder __________ ."

Set

Topic: Division of polynomials

Divide. You will need to draw your own boxes.

10 $(4x^3 - 3x^2 - 2x + 1) \div (x + 1)$

11. $(10x^3 - 26x^2 + 17z - 3) \div (5x - 3)$

13. $(2x^3 - 7x^2 + 9x - 4) \div (x - 1)$

14. $(x^3 - 17x + 4) \div (x - 4)$

Go

Topic: Features of polynomial functions

Write the key features of each function (intercepts, end behavior, and where the function is increasing/decreasing), then graph.

15. Equation: $f(x) = 9 - x^2$

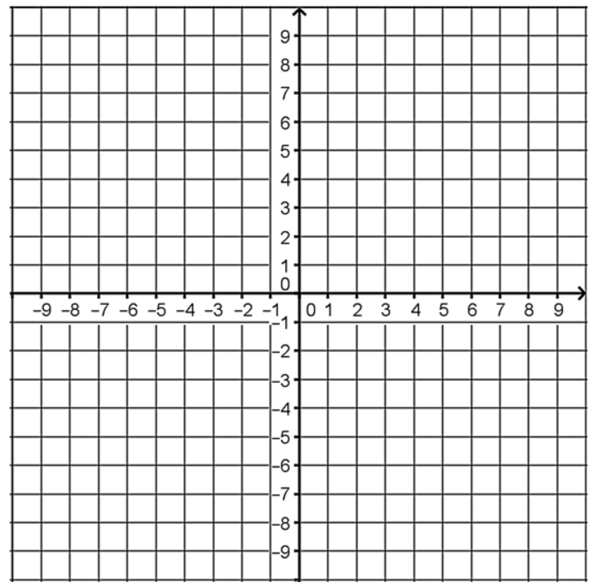
What I know about this function:

End behavior:

As $x \to -\infty$, $f(x) \to$ ______

As $x \to \infty$, $f(x) \to$ ______

Graph:



16. Equation: $h(x) = (x+1)\,(x-1)\,(x+2)$

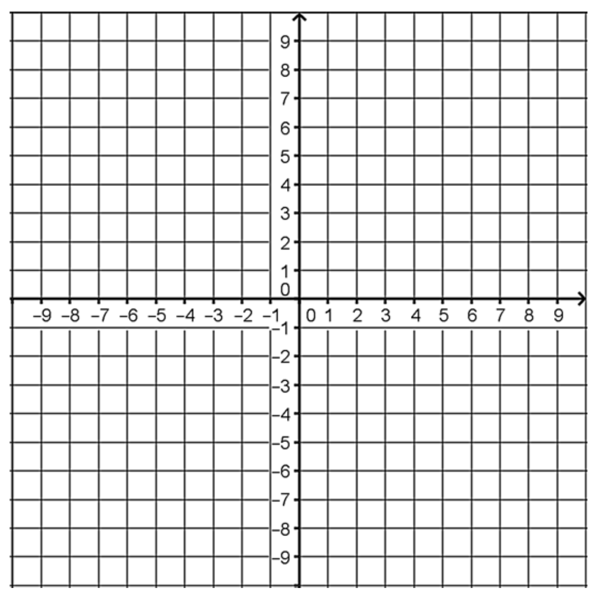
What I know about this function:

End behavior:

As $x \to -\infty$, $h(x) \to$ ______

As $x \to \infty$, $h(x) \to$ ______

Graph:



Name **Polynomial Functions** **3.7**

Ready, Set, Go!

Ready

Topic: Factoring special products

Factor.

1. $4x^2 - 25$
2. $9x^2 - 16y^2$
3. $a^2x^2 - b^2$

Factoring Rule for the Sum of Cubes: $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$

Factoring Rule for the Difference of Cubes: $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

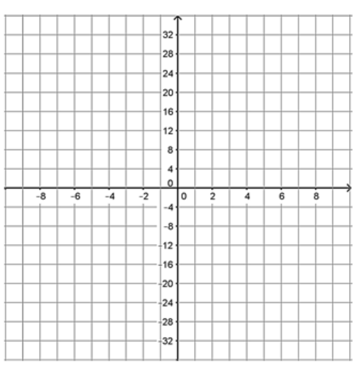
4. $64x^3 - 125$
5. $27x^3 + 8$
6. $1000x^3 - y^3$

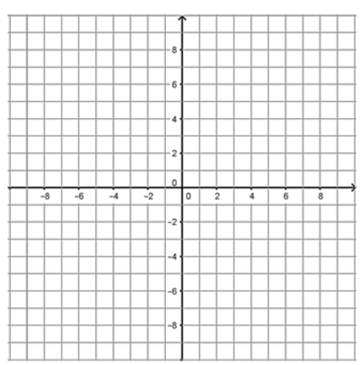
Set

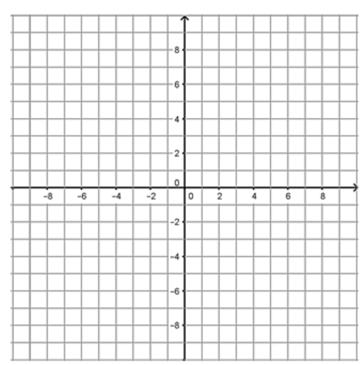
Topic: Finding zeros of polynomial functions.

Find all zeros of each polynomial, then sketch the graph. Use technology to check your answer.

7. $f(x) = x^2 - 25$
8. $g(x) = 4x^2 - 9$
9. $h(x) = x(x^2 - 5x + 6)$



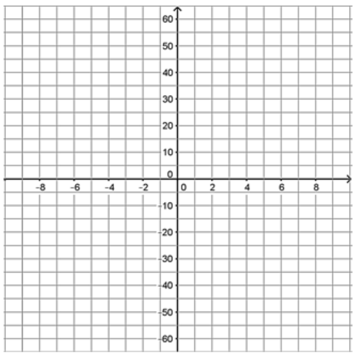


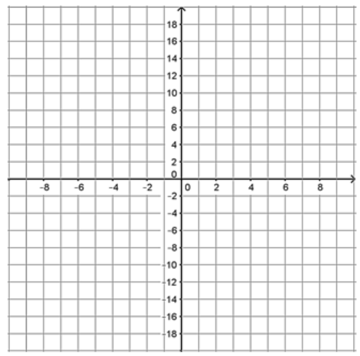


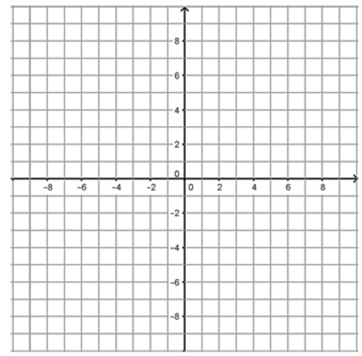
10. $f(x) = x^2 + 25$

11. $g(x) = 4x^2 + 9$

12. $h(x) = x(x^2 + 5x + 6)$







Topic: Using polynomial division

13. The product of two polynomials is $x^3 + 4x^2 + x - 6$. One of the factors is $x - 1$. Use the box method to find the other factors.

14. Use the box method to divide the following polynomials.
$(x^3 - 10x^2 + 29x - 56) \div (x - 7)$

Go

Topic: Multiply polynomials

Multiply each expression. Express your solutions in simplest form by combining like terms.

15. $(x-3)(x+3)$

16. $(x+4)(x+4)$

17. $(x-2)(x^2+2x+4)$

18. $(x+1)(x^2-x+1)$

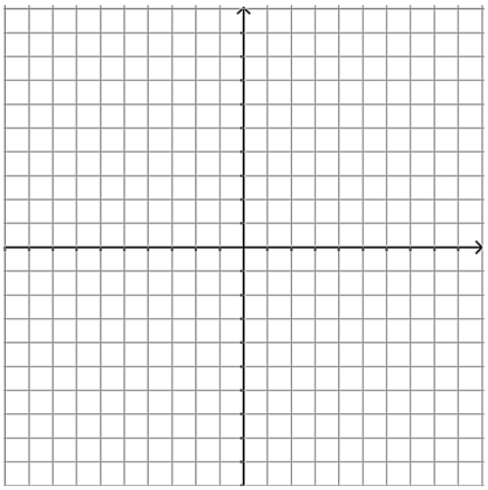
Ready, Set, Go!

Ready

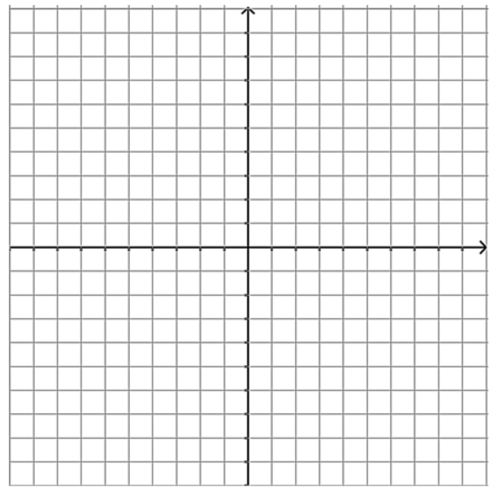
Topic: Graphing polynomial functions

Without using technology, sketch a graph of the polynomial function described (if possible). If not possible, state why not.

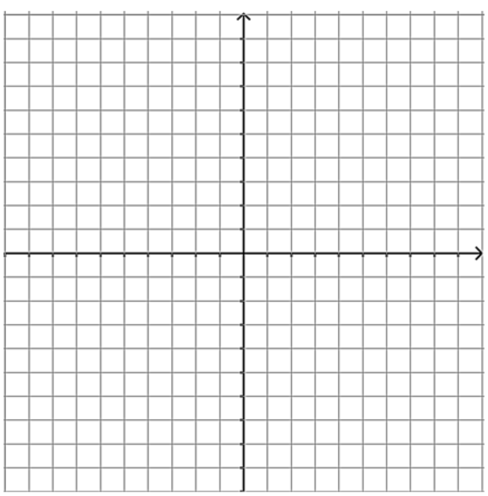
1. A cubic function with one negative zero (multiplicity 2) and one positive.



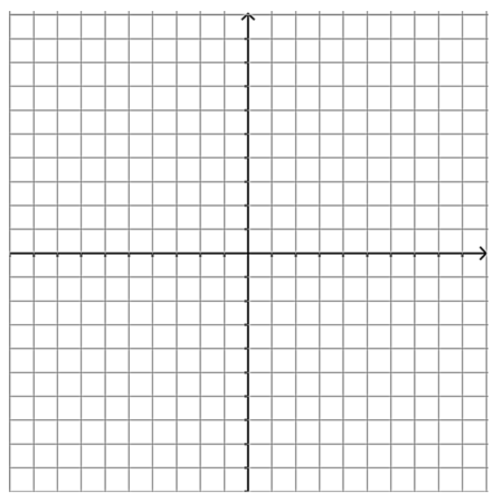
2. A quartic function (4th degree) with a negative leading coefficient, a positive y-intercept, one negative double root, one positive zero, and one additional zero.



3. A cubic function with zero real roots.



4. A quartic function with zero real roots, a positive leading coefficient, and a positive y-intercept.

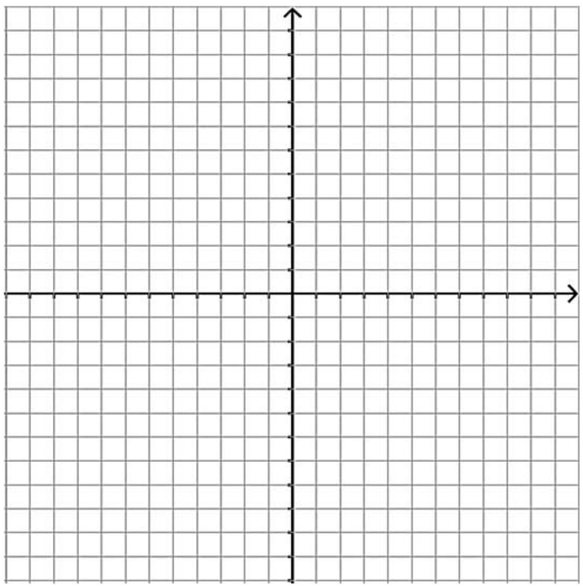


Set

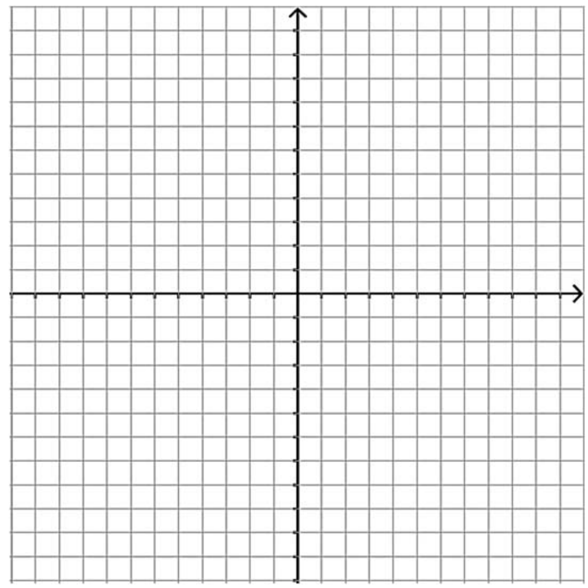
Topic: Finding factors of polynomial functions

Find all factors and sketch the graph of the polynomial functions (unless you see another method that allows for quicker graphing. If so, explain method).

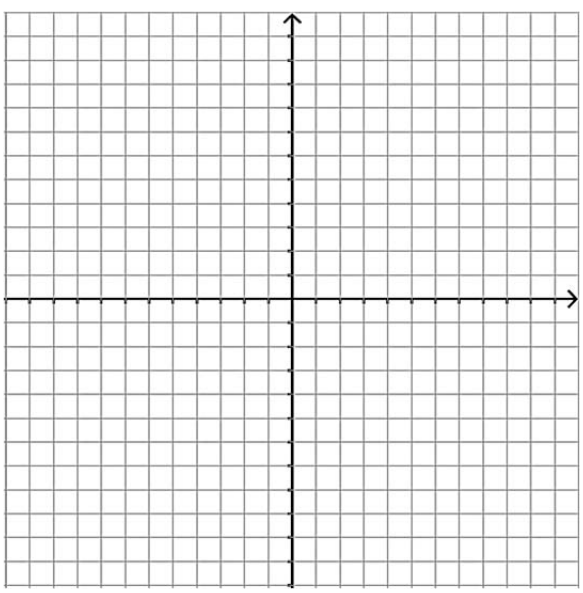
5. $f(x) = x^3 - 5x^2$



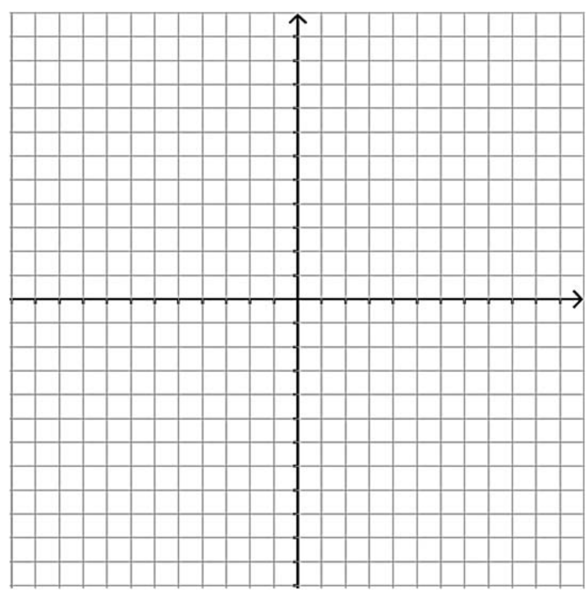
6. $f(x) = x^4 - 81$



7. $f(x) = x^3 - 5x^2 + 5x - 1$
   Hint: one root is $x = 1$

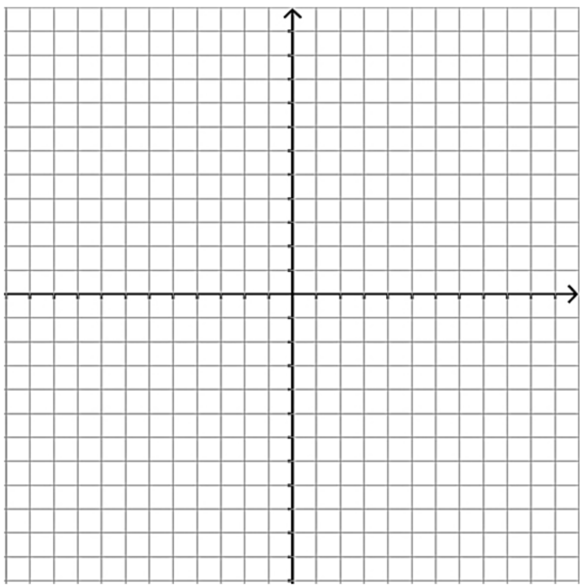


8. $f(x) = x^3 + 2x^2 + x + 2$
   Hint: one root is $x = -2$

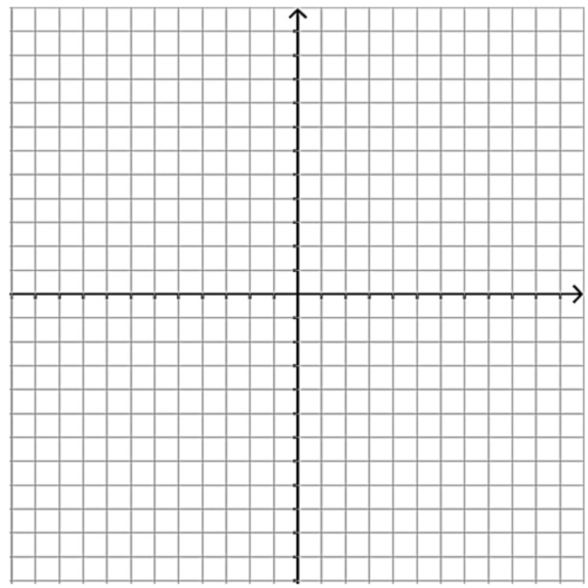


Use the Remainder Theorem to determine if the following are roots to the given equation. If so, find the other roots and graph the equation. Then write the function in factored form.

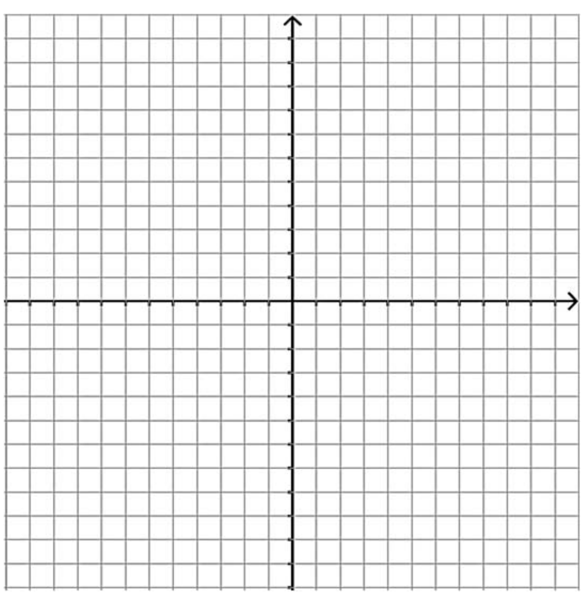
9. $f(x) = x^3 + 5x^2 + 2x - 8;\ f(1)$



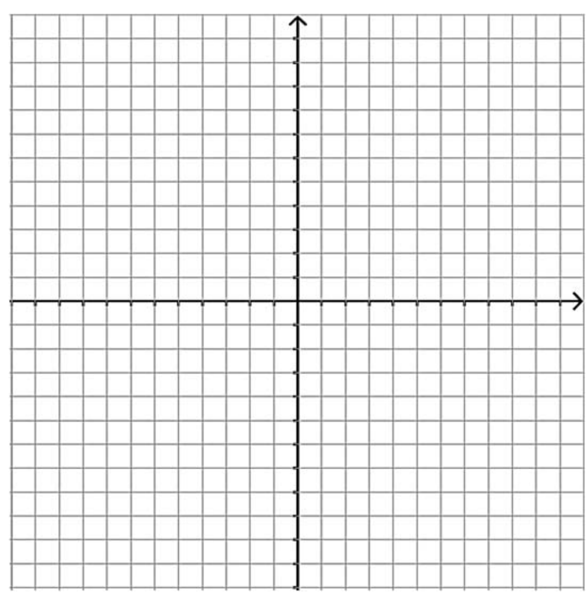
10. $f(x) = x^3 - 2x^2 + 9x - 10;\ f(2)$



11. $f(x) = x^3 - 19x - 30;\ f(-3)$



12. $f(x) = x^3 - x^2 - 2x;\ f(2)$



Go

Topic: Remainder Theorem and Fundamental Theorem of Algebra

13. In this Module, we have discussed the Fundamental Theorem of Algebra and the Remainder Theorem. Describe each theorem in your own words using diagram(s) to help illustrate your description.

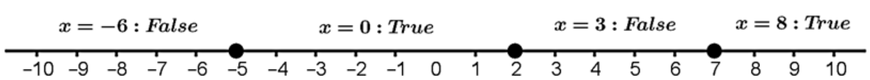
Go

Topic: Solving inequalities

Solve each inequality by placing the zeros of the related equation on a number line and checking a value in each interval. Express your solutions to the inequality in interval notation.

Example: $(x+5)(x-2)(x-7) \geq 0$

Related equation: $\boldsymbol{(x+5)(x-2)(x-7)=0}$
Zeros of the related equation: $\boldsymbol{-5, 2, 7}$



Solution to the inequality: $\boldsymbol{[-5, 2] \cup [7, \infty)}$

14. $x^2 + 7x + 6 < 0$

15. $3x - 5 > 2$

16. $(x+1)(x-1)(x-5) < 0$

17. $5x^2 - 17x + 14 > 0$

Name **Polynomial Functions** **3.9**

Ready, Set, Go!

Ready

Topic: Domain of a function

For each function below, state the domain of the function.

1. $f(x) = \sqrt{x+2}$
2. $h(x) = \log_2(x-5)$
3. $g(x) = \frac{1}{x} + 2$
4. $q(x) = \sqrt{1-x}$
5. $p(x) = 2 - \log_3 x$
6. $r(x) = \frac{15}{x+3}$

Set

Topic: Writing polynomial functions given roots

Write the polynomial function in standard form with least degree using the given information. Make sure to include any missing conjugate pairs.

7. Leading coefficient: 2; roots: $2, \sqrt{2}, -\sqrt{2}$
8. Leading coefficient: -1; roots: $1, 1+\sqrt{3}, 1-\sqrt{3}$
9. Leading coefficient: 2; roots: $4i$
10. Passes through the points $(2,0), (-3,0), (1,0), (0,1)$
11. Leading coefficient: 2; roots: $\sqrt{2}$ and 1
12. $f(1) = f(2) = f(-1) = f(-2) = 0, f(0) = -8$

Go

Topic: Expanding binomials

Use Pascal's triangle to help expand the following binomials.

13. $(2x-3)^4$

14. $(3a+2b)^3$

Topic: Finding roots of polynomial functions

Find the roots of the polynomial functions using the given information.

15. $f(x)=x^4+x^3-3x^2-x+2$, $x=1$ is a double root (multiplicity of 2)

16. $g(x)=x^3-7x^2+3x-21$, $g(7)=0$

Ready, Set, Go!

Ready

Topic: Solving polynomial, logarithmic, and rational equations.

Solve for *x*.

1. $2(x-2)(x+1)^2 = 0$
2. $\log_2 x = 4$
3. $x^3 - 1 = 0$
4. $x^2 + 4x - 9 = 0$
5. $\log_2 9 = x$
6. $\frac{3}{x} = 6$

Topic: Using the Remainder Theorem

Find $f(3)$ for each polynomial and state whether or not $(x-3)$ is a factor.

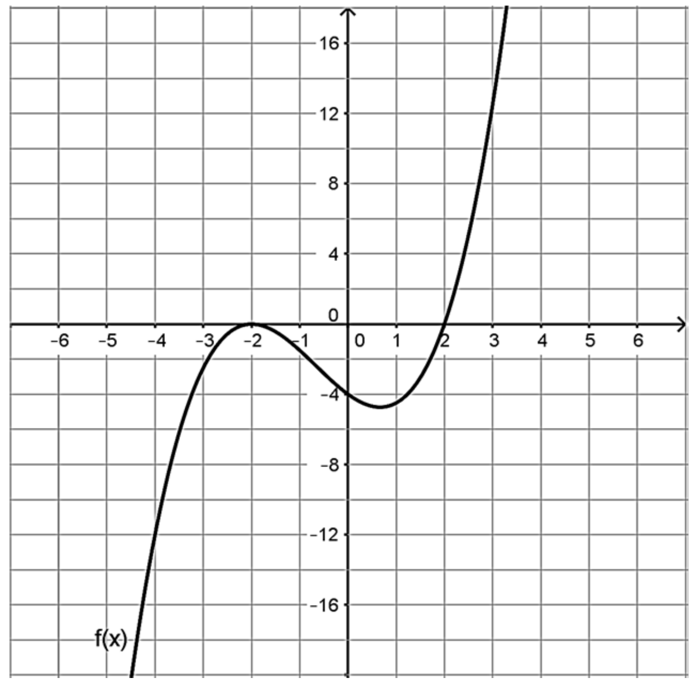
7. $f(x) = x^3 - 9x + 3$
8. $f(x) = x^3 - 9x^2 + 27x - 28$
9. $f(x) = 2x^3 - 5x^2 - 12x + 27$

Set

Topic: Graphing polynomial functions.

Complete the information below using the graph

10. Function (hint: the leading coefficient $\neq 1$):



End Behavior:

As $x \to -\infty,\ f(x) \to$ ____

As $x \to \infty,\ f(x) \to$ ____

Roots (with multiplicity):

Value of Leading Coefficient:

Domain:

Range:

11. Write the polynomial function with least degree, in both factored and standard forms, given the following roots and a point that the function passes through.

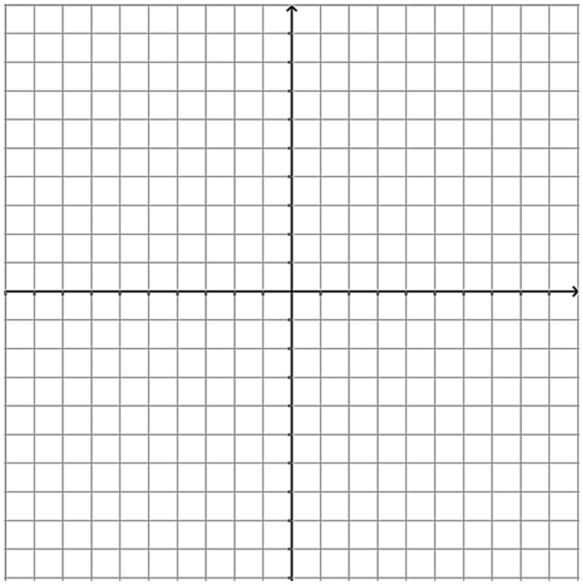
Roots: $\pm 1, 3$, Point on the graph: $(0, 9)$

Factored Form:

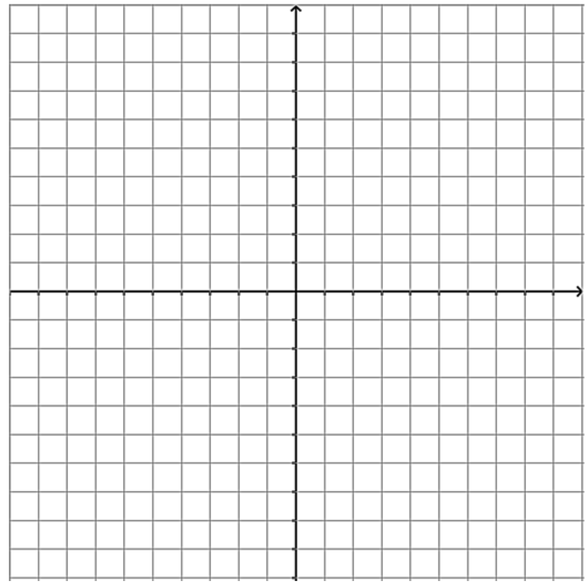
Standard Form:

Without using technology, sketch the graph of the polynomial function described.

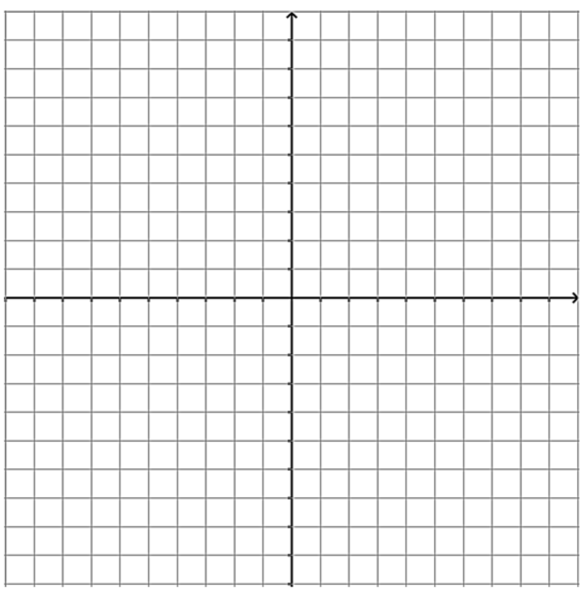
12. A cubic function with a leading coefficient of -1, with one positive zero.



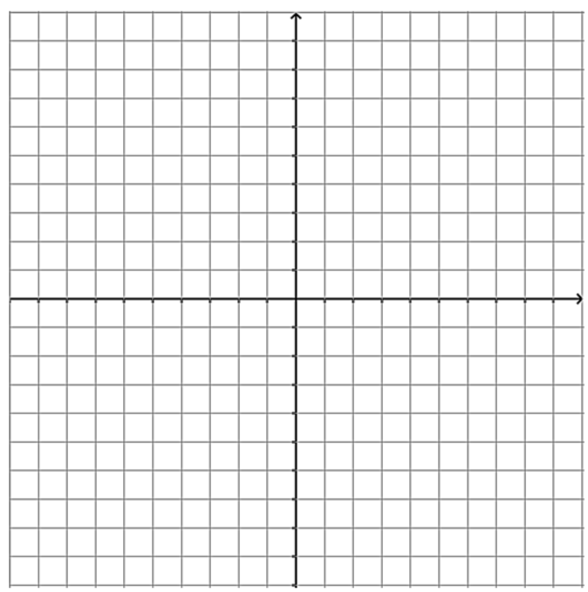
13. A quartic function with a leading coefficient of 1, with two double zeros.



14. A cubic function with a leading coefficient of -3, with one positive triple root.



15. A quartic function with a leading coefficient of 2, with two negative zeros and two imaginary roots.



Go

Topic: End behavior

Circle the expression that has the greatest value of $f(x)$ as $x \to \infty$.

16.	2^x	$x^2 - 2x + 10$	$x + 5$	$\log x$
17.	$\left(\frac{1}{2}\right)^x$	$x^2 - 2x + 10$	$x^5 - 4x^2$	$3\sqrt{x}7$
18.	$3 \cdot 2^x$	$x^3 + x^2 - 4$	$2(3^x)$	x^{10}

Integrated Math 3
Module 4
Rational Functions

Adapted from

The Mathematics Vision Project:
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Module 4 Overview

Prerequisite Concepts & Skills:

- Operations on polynomial expressions
- Graphing polynomial functions
- Translations of functions
- Graphing the following functions: linear, quadratic, exponential, logarithmic, square root, polynomial
- Features of the above functions: domain, range, intercepts, maxima/minima, discrete/continuous, end behavior, asymptotes
- Operations on rational numbers
- Polynomial division
- Proper vs. improper fractions
- Identifying extraneous solutions of equations

Summary of the Concepts & Skills in Module 4:

- Features of rational functions: domain, range, intercepts, vertical asymptotes, horizontal asymptotes, and end behavior
- Comparing rational numbers and rational expressions
- Operations on rational expressions
- Solving rational equations by finding the roots of the associated function
- Proper/improper rational expressions
- Identifying if rational functions are odd, even, or neither

Content Standards and Standards of Mathematical Practice Covered:

- Content Standards: F.IF.4, F.IF.5, F.IF.7d, F.IF.8, F.BF.3, A.SSE.1, A.APR.6, A.APR.7, A.REI.2, A.REI.11
- Standards of Mathematical Practice:
  1. Make sense of problems & persevere in solving them
  2. Reason abstractly & quantitatively
  3. Construct viable arguments & critique the reasoning of others
  4. Model with mathematics
  5. Use appropriate tools strategically
  6. Attend to precision
  7. Look for & make use of structure
  8. Look for & express regularity in repeated reasoning

Module 4 Vocabulary:

- Rational function
- Rational expression
- End behavior
- Vertical asymptote
- Horizontal asymptote
- Roots
- Domain
- Range
- Parent function
- Families of functions
- Proper/improper rational expression
- Degree
- Quotient
- Remainder
- Even function
- Odd function
- Sign line
- Curve sketch
- Multiplicity
- Inverse variation
- Extraneous solutions

Concepts Used in the Next Module:

- Surface Area
- Volume
- Area
- Figure Dissections
- Solving Right Triangles
- Pythagorean Theorem

Module 4 – Rational Functions

4.1 *Connecting rational expressions to rational numbers, performing operations on rational expressions.* **(A.APR.7)**
Warm Up: Polynomial Expressions
Classroom Task: What Does it Mean to be Rational? – A Develop Understanding Task
Ready, Set, Go Homework: Rational Functions 4.1

4.2 *Solving rational equations.* **(A.CED.2)**
Warm Up: Solve Me
Classroom Task: What's Your Solution? – A Develop Understanding Task
Ready, Set, Go Homework: Rational Functions 4.2

4.3 *Using context to identify inverse variation and introduce rational functions* **(F.IF.7d, A.CED.2, F.IF.5)**
Warm Up: Fraction Frenzy
Classroom Task: The Gift – A Develop Understanding Task
Ready, Set, Go Homework: Rational Functions 4.3

4.4 *Analyzing the characteristics of various families of functions to assist in identifying characteristics of rational functions.* **(A.SSE.1, F.IF.4, F.IF.7d, F.IF.8, F.BF.3)**
Warm Up: Begin Task
Classroom Task: All in the Family– A Solidify Understanding Task
Ready, Set, Go Homework: Rational Functions 4.4

4.5 *Connecting rational numbers: improper fractions to rewrite improper rational* **(A.APR.6)**
Warm Up: Proper v. Improper Fractions
Classroom Task: Rewriting Rational Expressions – A Solidify Understanding Task
Ready, Set, Go Homework: Rational Functions 4.5

4.6 *Identifying the end behavior of rational functions.* **(F.IF.4, F.IF.7d)**
Warm Up: End Behavior
Classroom Task: Watch Your Behavior – A Develop Understanding Task
Ready, Set, Go Homework: Rational Functions 4.6

4.7 *Determining if functions are even or odd* **(F.BF.3)**
Warm Up: Reflections
Classroom Task: Even/Odd Function Recognition – A Develop Understanding Task
Ready, Set, Go Homework: Rational Functions 4.7

4.8 *Graphing rational functions using features of rational functions* **(F.IF.4, F.IF.7d)**
Warm Up: Rational Function Graphs and their Features
Classroom Task: Features of Rational Functions – A Solidify Understanding Task
Ready, Set, Go Homework: Rational Functions 4.8

4.9 *Graphing rational functions* **(F.IF.4, F.IF.7d, A.REI.2, A.REI.11)**
Classroom Task: Graphing Rational Functions – A Practice Understanding Task
Ready, Set, Go Homework: Rational Functions 4.9

4.1 Warm Up
Polynomial Expressions

Simplify each expression completely.

1. $3(x-8)+2(x+5)$

2. $4(x+6)-2(x+8)$

3. $\frac{18(x-6)(x+2)}{24(x^2-10x+24)}$

Factor each expression completely.

4. $x^2+5x-24$

5. $2x^2+17x+21$

4.1 What Does it Mean to Be Rational?
A Develop Understanding Task

Part I: Comparing rational numbers and rational expressions.

1. In your own words, define ***rational number***.

   Circle the numbers below that are rational and refine your definition, if needed.

   3 $\quad -5$ $\quad \frac{2}{3}$ $\quad \frac{20}{3}$ $\quad 14$ $\quad 2.7$ $\quad \sqrt{5}$ $\quad 2^3$ $\quad 3^{-3}$ $\quad \log_2 8$ $\quad \frac{7}{0}$

2. The definition of a ***rational function*** is as follows:

   A function, $f(x)$, is called a ***rational function*** if and only if it can be written in the form $f(x) = \frac{P(x)}{Q(x)}$ where P and Q are polynomials in terms of x and Q is not the zero polynomial.

   Restate the definition in your own words and then write three examples of rational functions.

3. How are rational numbers and rational expressions similar? Different?

Part II: Arithmetic of rational expressions: making connections between rational numbers and rational expressions.

Solve problems in the left column and then use a similar process to simplify the rational expressions in the center column. Leave answers in the center column in factored form whenever possible.

Arithmetic of Rational Numbers	Arithmetic of Rational Expressions	Explain the Method Used for each Rational Expression
4a. $\frac{2}{3}+\frac{4}{7}$	4b. $\frac{3}{(x+1)}+\frac{4x}{(x+3)}$	
5a. $\frac{3}{8}+\frac{5}{6}$	5b. $\frac{2x}{(x+3)(x+2)}+\frac{4x}{(x-1)(x+3)}$	
6a. $\frac{7}{8}-\frac{1}{6}$	6b. $\frac{2x}{(x+3)}-\frac{4}{(x-1)}$	
7a. $\frac{3}{8}\cdot\frac{5}{6}$	7b. $\frac{(x+1)(x-2)}{(x+2)}\cdot\frac{(x+5)}{(x-2)(x+2)}$	
8a. $\frac{3}{8}\div\frac{5}{6}$	8b. $\frac{(x+1)(x-2)}{(x+2)}\div\frac{(x+5)}{(x-2)(x+2)}$	

Practice Problems:

Simplify each expression as much as possible.

1. $\frac{x^2+12x+20}{3x+2}$

2. $\frac{6}{x^2-9x+20} \cdot \frac{5x-25}{15}$

3. $\frac{5}{x+1} \div \frac{3x-21}{x^2-6x-7}$

4. $\frac{9x+18}{x^2-2x-8} \div \frac{6x}{3x-12}$

5. $\frac{3}{y+5} + \frac{y}{y^2+7y+10}$

6. $\frac{2x+3}{5x-30} - \frac{3x+4}{x-6}$

7. $\frac{6x-7}{x^2+6x+5} - \frac{4}{x+5}$

8. $\frac{x^2-5x-6}{2x+6} \div \frac{x^2-3x-4}{4x+12}$

4.2Warm Up
Solve Me

Solve each equation.

1. $\frac{2x-5}{3x} = \frac{9}{8}$

2. $\frac{1}{3}x + \frac{3}{4} = \frac{7}{6}x - \frac{2}{9}$

3. $\frac{x+7}{2} = \frac{3x+1}{5}$

4. $\frac{5}{2} - \frac{7}{8}x = \frac{9}{4}x + 5$

4.2 What's Your Solution?
A Develop Understanding Task

1. In the previous task, you developed the idea of finding roots of a function. How are the roots of a function related to its graph?

2. Describe the process for finding roots of any function.

Below are more complicated examples of rational equations. Find the all values of *x* that make the equation true. Be sure to check for extraneous solutions.

3. $1 - \frac{1}{x-1} = 0$

4. $2 = \frac{5}{x^2+2}$

5. $\frac{1}{x^2-5x} = \frac{x+7}{x} - 1$

6. $\frac{x}{x-2} + \frac{1}{x-4} = \frac{2}{x^2-6x+8}$

7. $x + \frac{6}{x-3} = \frac{2x}{x-3}$

8. $\frac{x}{x-1} + \frac{3}{x} = \frac{5}{2x}$

Practice Problems:

Solve each equation. Be sure to look for extraneous solutions.

1. $1 - \frac{8}{x-5} = \frac{3}{x}$

2. $\frac{x}{x+1} + \frac{2}{x+4} = 1$

3. $\frac{6}{x-3} = \frac{8x^2}{x^2-9} - \frac{4x}{x+3}$

4. $\frac{1}{2x} + \frac{3}{x+7} = -\frac{1}{x}$

5. $\frac{5}{x^2+x-6} = 2 + \frac{x-3}{x-2}$

6. $\frac{x+3}{x-3} + \frac{x}{x-5} = \frac{x+5}{x-5}$

4.3 Warm Up
Fraction Frenzy

Simplify each expression.

1. $\frac{12}{x^2+5x-24}+\frac{3}{x-3}$

2. $\frac{x^2-3x-10}{x^2+4x+3}\cdot\frac{x^2+2x-3}{x^2+x-2}$

3. $\frac{x^2-8x+15}{x^2+4x}\div(x^2-x-20)$

4. $\frac{x+4}{x^2-4}-\frac{15}{x-2}$

Solve each equation.

5. $\frac{3x}{x+1}+\frac{6}{2x}=\frac{7}{x}$

6. $\frac{1}{x+6}+\frac{x+1}{x}=\frac{13}{x+6}$

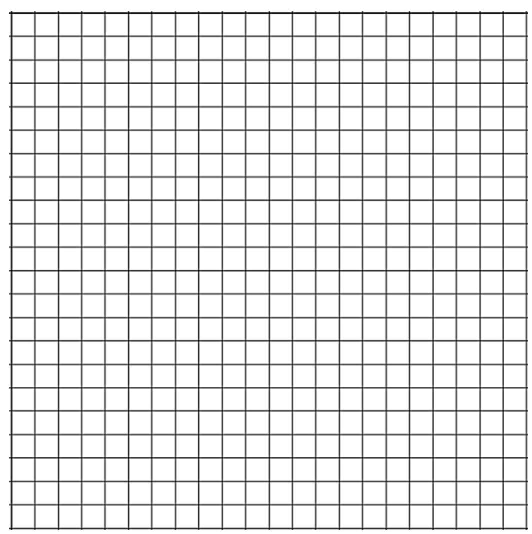
4.3 The Gift
A Develop Understanding Task

Chile is celebrating her Quinceañera. Hannah knows the perfect gift to buy Chile, but it costs \$360. Hannah can't afford to pay for this on her own so thinks about asking some friends to join in and share the cost.

1. How much would each person spend if there were two people dividing the cost of the gift equally? How much would each person spend if there were three people dividing the cost equally? Five people? Ten? One hundred?

2. Determine the function that could be used to model the amount each person would spend depending on the number of people contributing to the gift.

3. Use multiple representations to show how the amount each person would contribute to the gift would change depending on the number of people contributing. Describe the connections between the representations.



4. Describe the features of the function based on the **context** (domain/range, increasing/decreasing, maxima/minima, discrete/continuous, end behavior, intercepts, asymptotes).

5. Kristina is taking a 100 mile day trip with her family.
   a. How long will the trip take if Kristina's family averages 30 miles per hour? 55 miles per hour? or 65 miles per hour?

   b. Write a function that describes the time it takes to make this trip as a function of the car's speed. Identify the meaning of the variables.

   c. Use the values from part a as well as some additional values to complete the table. Then graph this function.

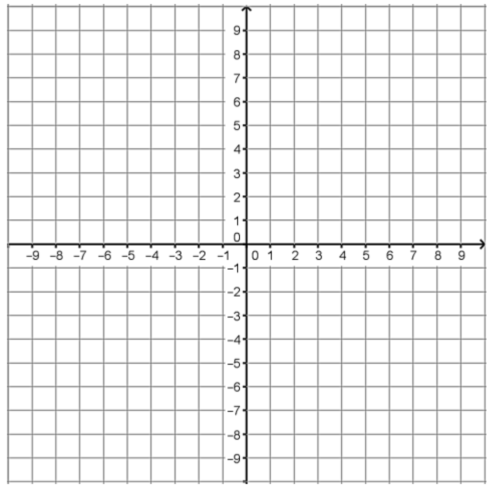
   d. It is said that the time it takes for Kristina's family to travel **varies inversely** with the average speed of the car. Describe what you think this means in your own words.

Practice Problems:

For each function, fill in the table of values and then graph the function. Then list the features of the function (domain/range, continuous/not continuous, intercepts, etc.).

1. $f(x) = \frac{1}{x}$

x	$f(x)$
-2	
-1	
$-\frac{1}{2}$	
0	
$\frac{1}{2}$	
1	
2	



List of Features:

Domain:

Range:

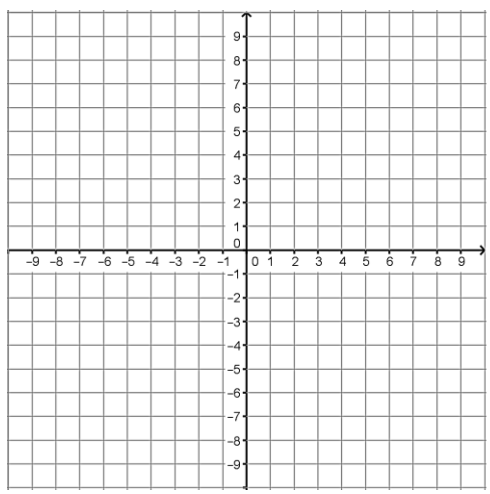
Continuous:

Intercepts:

End behavior:

2. $f(x) = \frac{1}{x-4}$

x	$f(x)$
1	
2	
$3\frac{1}{2}$	
4	
$4\frac{1}{2}$	
5	
6	



List of Features:

Domain:

Range:

Continuous:

Intercepts:

End behavior:

4.4 All in the Family
A Develop Understanding Task

We have studied several families of functions including linear, exponential, quadratic, logarithmic, square root, and polynomial. In this task, we will examine features of families of functions from our previous work and also look at the features of ***rational functions***.

Part I: Finding Features.

Describe the process you use to find the given features of specific functions. Then generalize this process to find the given features of **any** function.

1. The process I use to find **roots** for the following functions:

Linear	Logarithmic	Polynomial (in factored form)

In general, you find the **roots** of a function by…

2. The process I use to determine **end behavior** for the following functions:

Quadratic	Exponential	Polynomial

In general, you determine the **end behavior** of a function by…

3. **Asymptotes** occur when…

Logarithmic	Exponential

In general, **asymptotes** of a function occur when…

Explain how to find the **asymptotes**.

4. The **domain** of a function is…

Square Root	Logarithmic	Polynomial

In general, you find the **domain** of a function by…

Part II: Characteristics of Rational Functions

In the task, *The Gift,* we saw a rational function used to model the situation with Chile's Quinceañera. ***Rational functions*** are any function, $f(x)$, that can be written as the ratio between two polynomial functions. That is, $f(x) = \frac{P(x)}{Q(x)}$, where P and Q are polynomial functions in terms of x and Q is not the zero polynomial function.

Below are examples of rational functions. Like other functions we have studied, rational functions come in different forms. Each form highlights different aspects of the function.

a. $f(x) = \frac{1}{x}$ b. $g(x) = \frac{(x-3)(x+1)}{x(x-1)(x-4)}$ c. $h(x) = \frac{x^2+3x-18}{x^4+4x^2-5}$

5. Which functions(s) allow us to easily identify the roots? Explain.

6. Which function(s) allow us to easily identify the vertical asymptotes? Explain.

7. Which function(s) allow us to easily identify the end behavior? Explain.

8. Based on other functions we have studied, state how you would find the following features of a rational function. Then find those features using functions b and c from above.

State how you would identify each feature of a rational function:	**Find the following features using function b: $g(x) = \frac{(x-3)(x+1)}{x(x-1)(x-4)}$**	**Find the following features using function c: $h(x) = \frac{x^2+3x-18}{x^4+4x^2-5}$**
Roots:	Roots:	Roots:
Vertical Asymptotes:	Vertical Asymptotes:	Vertical Asymptotes:
End Behavior:	End Behavior:	End Behavior:

4.5 Warm Up
Proper v. Improper Fractions

Classify each fraction as proper or improper. If the fraction is improper, rewrite as a mixed number.

1. $\frac{27}{5}$
2. $\frac{5}{27}$
3. $\frac{32}{14}$
4. $\frac{315}{123}$
5. $\frac{14}{9}$
6. $\frac{13x^3}{18x^5}$

4.5 Rewriting Rational Expressions
A Solidify Understanding Task

Part I: Comparing proper fractions and proper rational expressions (as well as improper).

1. What is the difference between a proper fraction and an improper fraction?

Rational expressions are similar to rational numbers, except that instead of comparing the numeric value of the numerator and denominator, the comparison is based on the degree of each polynomial. ***A rational expression is proper if the degree of the numerator is less than the degree of the denominator, and improper otherwise.*** A rational expression, $\frac{a(x)}{b(x)}$, where $a(x)$ and $b(x)$ are polynomials, ***is improper if the degree of $a(x)$ is greater than or equal to the degree of $b(x)$.***

2. Label each rational expression as proper or improper.

$\frac{(x+1)}{(x-2)(x+2)}$ $\quad$ $\frac{x^3-3x^2+5x-1}{x^2-4x+4}$ $\quad$ $\frac{(x+3)(x+2)}{x^4-4}$ $\quad$ $\frac{x+3}{x+5}$ $\quad$ $\frac{x^3-5x+2}{x-10}$

As we may remember, improper fractions can be rewritten in an equivalent form we call a mixed number. If $a > b$, then the fraction $\frac{a}{b}$ can be rewritten as $\frac{a}{b} = q + \frac{r}{b}$, where q represents the quotient and r represents the remainder.

Rewrite each improper fraction as an equivalent mixed number.

3. $\frac{35}{5} =$ $\quad$ 4. $\frac{37}{5} =$ $\quad$ 5. $\frac{247}{12} =$

Part II:

Determine if each rational expression is proper or improper. If improper, divide the polynomials to rewrite the rational expressions such that $\frac{a(x)}{b(x)} = q(x) + \frac{r(x)}{b(x)}$ where $q(x)$ represents the quotient and $r(x)$ represents the remainder.

6. $\frac{x^2+5x+6}{x+2}$

7. $\frac{3x-4}{x^3-1}$

8. $\frac{2x^3-7x^2+6}{x-1}$

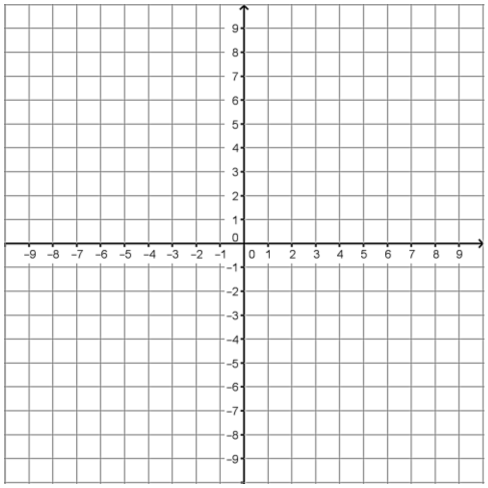
9. $\frac{-5x+10}{x^3+6x^2+3x-1}$

10. $\frac{x^2+2x+5}{x+3}$

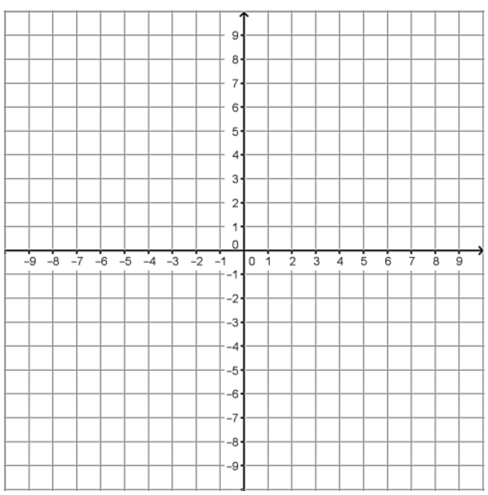
11. $\frac{3x+8}{x-1}$

12. $\frac{4x-1}{x+2}$

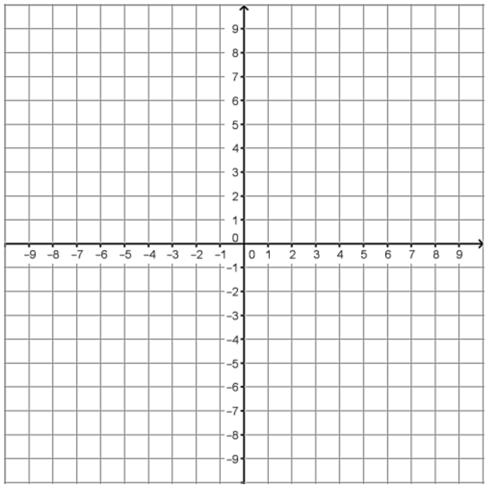
13. Using technology, graph the rational function from question 6, $f(x) = \frac{x^2+5x+6}{x+2}$ and sketch it on the grid below. How does the graph relate to the answer from question 6?



14. Using technology, graph the rational function from question 11, $g(x) = \frac{3x+8}{x-1}$ and sketch it on the grid below. How does the graph relate to the transformation of the parent function, $y = \frac{1}{x}$?



15. Using technology, graph the rational function from question 12, $h(x) = \frac{4x-1}{x+2}$ and sketch it on the grid below. How does the graph relate to the transformation of the parent function, $y = \frac{1}{x}$?



4.6 Warm Up
End Behavior

Based on the equations alone, describe what happens as $x \to \infty$ and $x \to -\infty$.

1. $h(x) = 2x - 3x^2 + 7$

   as $x \to \infty, h(x) \to$

   as $x \to -\infty, h(x) \to$

2. $g(x) = 14x^5 - 100x^4 + 1$

   as $x \to \infty, g(x) \to$

   as $x \to -\infty, g(x) \to$

3. $f(x) = 3x - 4 + x^6$

   as $x \to \infty, f(x) \to$

   as $x \to -\infty, f(x) \to$

4.6 Watch Your "Behavior"
A Develop Understanding Task

In this task, you will develop your understanding of the **end behavior** of rational functions.

After completing the task, *The Gift*, Marcus and Hannah were talking about the discussion regarding the end behavior of the parent function $f(x) = \frac{1}{x}$. Marcus said "I thought the end behavior of all functions was that you either ended up going to positive or negative infinity." Hannah agreed, adding, "Now we have a function that approaches zero. I wonder if all rational functions will always approach zero as *x* approaches $\pm\infty$." Marcus replied "I am sure they do. Just like all polynomial functions end behavior approaches either $\pm\infty$, I think the end behavior for all rational functions must approach zero".

1. Analyze the end behavior of different **proper** and **improper** rational functions using a graphing utility. Record the functions and end behaviors in the table and try to generalize the patterns you notice regarding end behavior of proper rational functions.

Rational Function	**End Behavior**
Proper Rational Functions:	
Improper Rational Functions with Equal Degrees in the Numerator and Denominator:	
Improper Rational Functions where the Numerator's Degree is Greater than the Denominator's Degree:	

2. Marcus stated: "just like all polynomial functions end behavior approaches either $\pm\infty$, I think the end behavior for all rational functions must approach zero." Is he always correct, sometimes correct, or not correct? Explain thoroughly.

3. Describe the end behavior of $f(x) = \frac{2x-3}{x+1}$.

   As $x \to \infty, f(x) \to$ ______

   As $x \to -\infty, f(x) \to$ ______

4. Simplify the rational expression $\frac{2x-3}{x+1}$ using division. Write the answer in the form $\frac{a(x)}{b(x)} = q(x) + \frac{r(x)}{b(x)}$.

5. a. As *x* approaches infinity, what happens to the value of your quotient, $q(x)$, found in question 4?

   b. What happens to the value of your quotient as *x* approaches negative infinity?

   c. What happens to the remainder, $\frac{r(x)}{b(x)}$, as *x* approaches positive or negative infinity?

6. Repeat the process you went through in questions 3-5 for the function $g(x) = \frac{3x^2+15x+18}{2x^2-32}$.

   a. Describe:

   b. Simplify using division:

   c. What happens to the quotient, $q(x)$, and the remainder, $\frac{r(x)}{b(x)}$, as $x \to \pm\infty$

7. Repeat the process again for the function $h(x) = \frac{x^3+2x+1}{x-3}$.

   a. Describe:

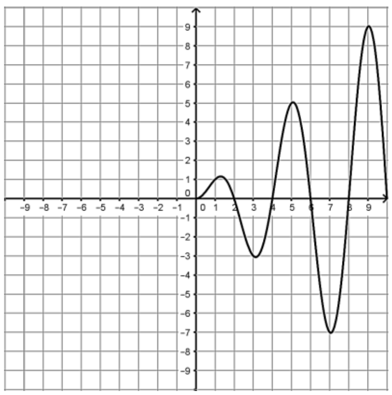
   b. Simplify using division:

   c. What happens to the quotient, $q(x)$, and the remainder, $\frac{r(x)}{b(x)}$, as $x \to \pm\infty$

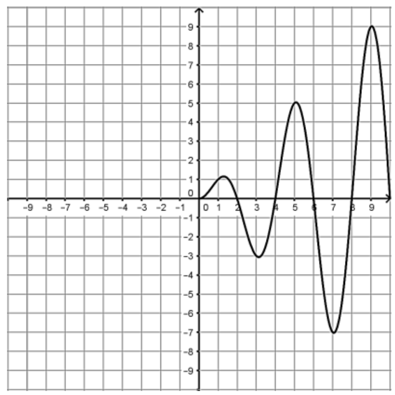
4.7 Warm Up
Reflections

Sketch the reflection of each graph across the given line of reflection.

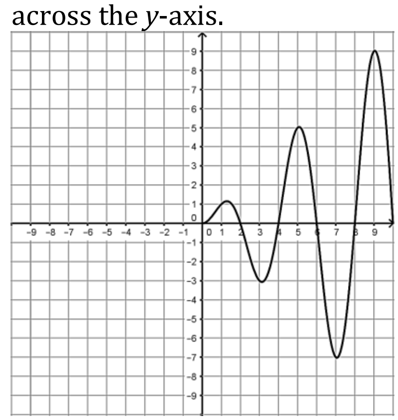
1. Reflect across the *x*-axis.



2. Reflect across the *y*-axis.

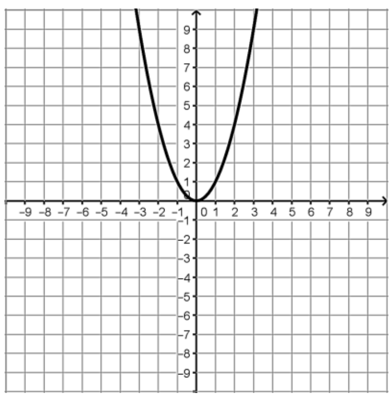
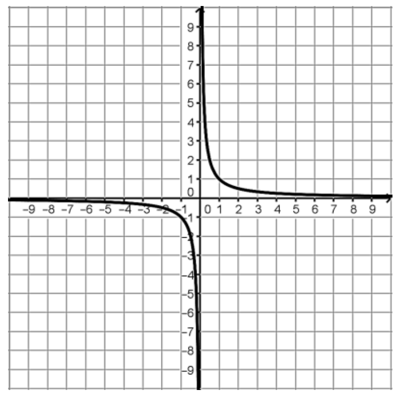
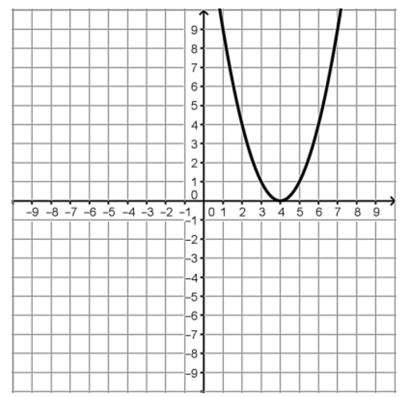
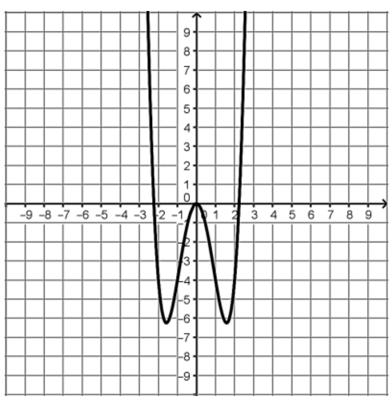
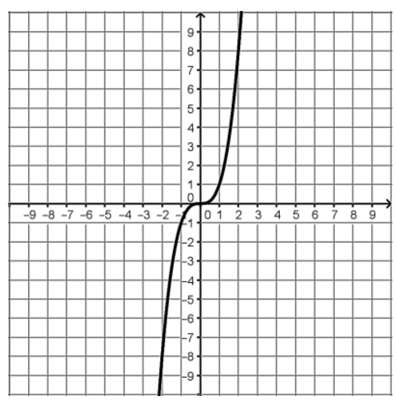
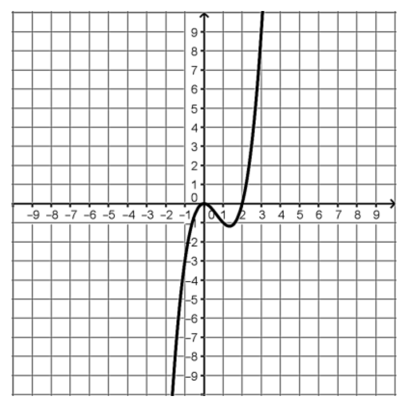


3. Reflect across the *x*-axis, then across the *y*-axis.



4.7 Even/Odd Function Recognition
A Develop Understanding Task

Below are six graphs: two that represent even functions, two odd functions, and two functions that are neither even nor odd.

Even function: $f(x) = x^2$ 	Odd function: $f(x) = \frac{1}{x}$ 	Neither: $f(x) = (x-4)^2$ 
Even function: $f(x) = x^4 - 5x^2$ 	Odd function: $f(x) = x^3$ 	Neither: $f(x) = x^3 - 2x^2$ 

1. What symmetries do you notice for an **even** function?

2. What symmetries do you notice for an **odd** function?

3. a. Evaluate each function at the given values of x.

Even functions:

$f(x) = x^2$ $(2, \_)$ $(-2, \_)$
$(5, \_)$ $(-5, \_)$

$f(x) = x^4 - 5x^2$ $(2, \_)$ $(-2, \_)$
$(5, \_)$ $(-5, \_)$

Odd functions:

$f(x) = \frac{1}{x}$ $(2, \_)$ $(-2, \_)$
$(5, \_)$ $(-5, \_)$

$f(x) = x^3$ $(2, \_)$ $(-2, \_)$
$(5, \_)$ $(-5, \_)$

b. What do you notice about the corresponding $f(x)$ values for the **even** functions?

c. What do you notice about the corresponding $f(x)$ values for the **odd** functions?

4. Use the graphs and their corresponding functions to write a definition for an even function and an odd function.

a. A function is an **even** function if…

b. A function is an **odd** function if …

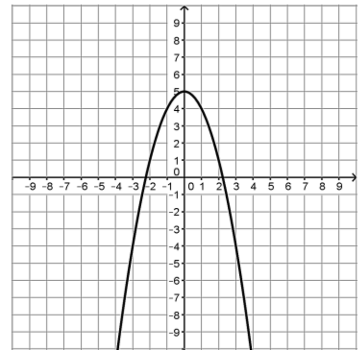
5. Below are more functions. Based on your definition in question 4, classify the following functions as either even, odd, or neither. Explain your reasoning for each problem.

a.

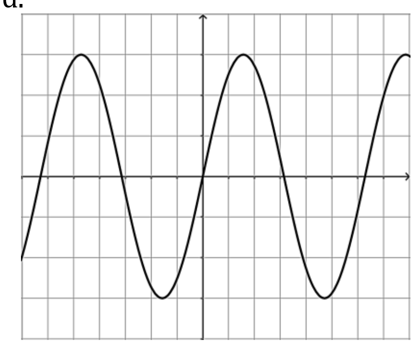
x	y
-2	10
-1	5
0	-4
1	5
2	10

b. $f(x) = x^4 - 3x + 6$

c.



d.



e. $f(x) = x^4 - 3$

f. $f(x) = x^5$

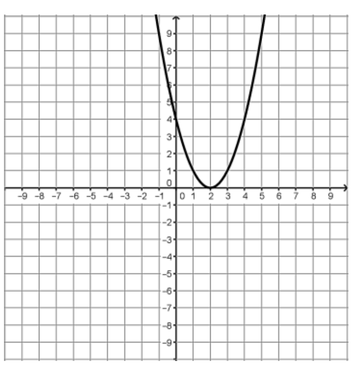
g.

x	y
-2	-10
-1	-5
0	0
1	5
2	10

h. $f(x) = \sqrt{x+2}$

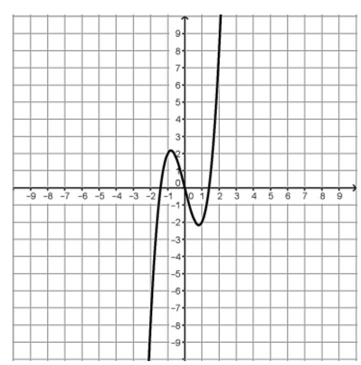
i. $f(x) = x(x-2)(x+2)$

j.



k. $f(x) = |x| - 3$

l.



6. Ask your teacher for a copy of the answers to question 5. Check your solutions and adjust your definitions in question 4 of even and odd functions.

4.8 Warm Up
Rational Function Graphs and their Features

$f(x) = \frac{x+1}{(x-2)(x+2)}$

x-intercept(s):

y-intercept:

Vertical asymptote(s):

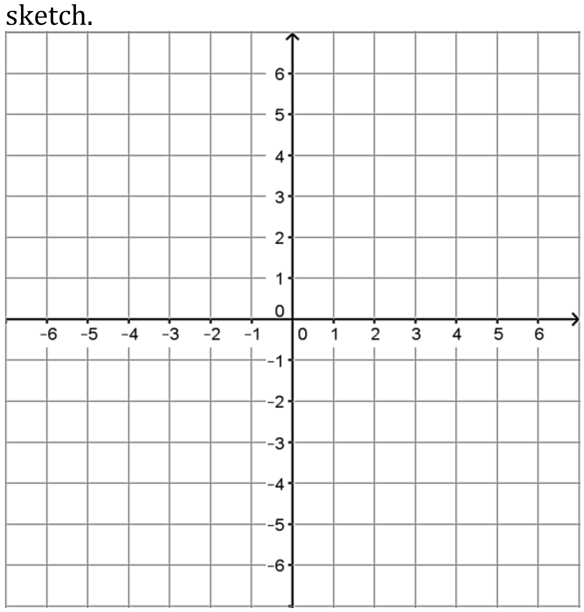
Proper or Improper:

End behavior:

Equation of any horizontal asymptotes:

Create a sign line using x-intercepts and values of vertical asymptotes. Test values in each region of the sign line and compare to the value of the horizontal asymptote.

Sketch the features on the graph below. Then use a table of values or graphing utility to complete the sketch.



Why would we use the x-intercepts and values of the vertical asymptotes in our sign lines?

4.8 Features of Rational Functions
A Solidify Understanding Task

Part I:

Using prior knowledge, determine the features of each rational function and then sketch a graph.

1. $f(x) = \frac{x^2-4}{(x+5)(x+1)}$

x-intercept(s):

y-intercept:

Vertical asymptote(s):

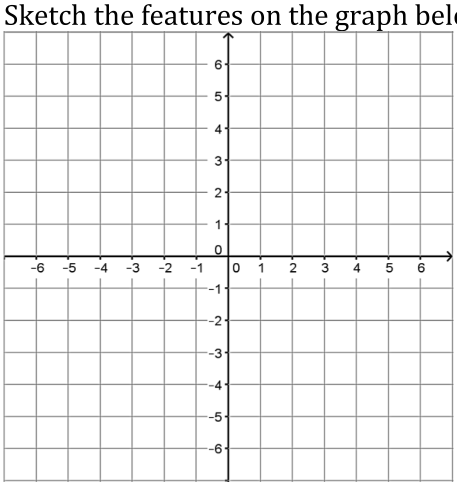
Proper or Improper:

End behavior:

Equation of any horizontal asymptotes:

Create a sign line using x-intercepts and values of vertical asymptotes. Test values in each region of the sign line and compare to the value of the horizontal asymptote.

Sketch the features on the graph below:



2. $f(x) = \frac{1}{(x+3)(x+1)}$

x-intercept(s):

y-intercept:

Vertical asymptote(s):

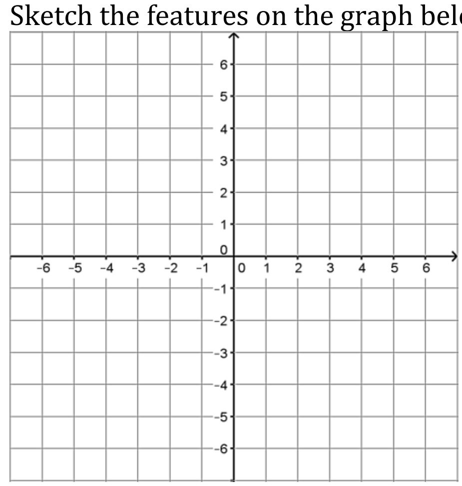
Proper or Improper:

End behavior:

Equation of any horizontal asymptotes:

Create a sign line using x-intercepts and values of vertical asymptotes. Test values in each region of the sign line and compare to the value of the horizontal asymptote.

Sketch the features on the graph below:



3. Describe the end behavior and any horizontal asymptotes for rational functions falling into the following categories:

   a. The numerator and denominator have equal degrees.

   b. The degree in the numerator is greater than the degree in the denominator.

   c. The degree in the numerator is less than the degree in the denominator.

4. Complete the sentence: The domain of a rational function is all real numbers except…

Part II:

Find the roots of the function, if any exist. Identify the vertical and horizontal asymptotes, complete a sign line, and sketch a graph of the function.

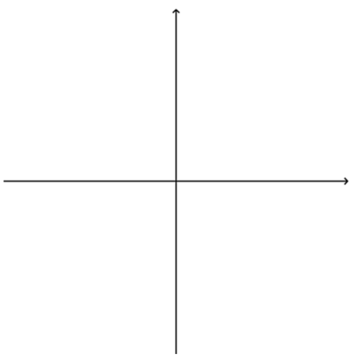
5. $f(x) = \frac{-3}{x^2-3x+2}$

Roots:

Vertical Asymptotes:

Horizontal Asymptotes:

Sign Line:



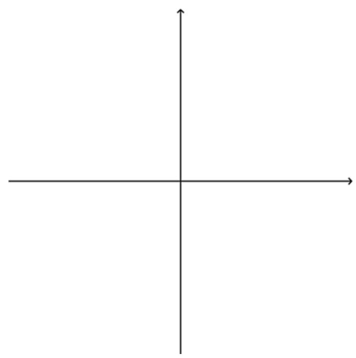
6. $f(x) = \frac{2x-5}{x+2}$

Roots:

Vertical Asymptotes:

Horizontal Asymptotes:

Sign Line:



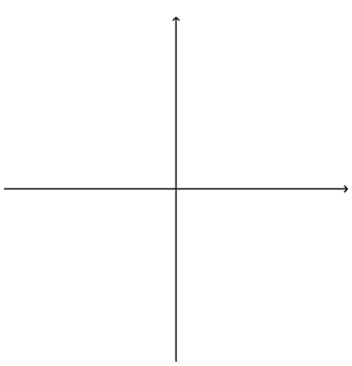
7. $f(x) = \frac{-x}{x^2-16}$

Roots:

Vertical Asymptotes:

Horizontal Asymptotes:

Sign Line:



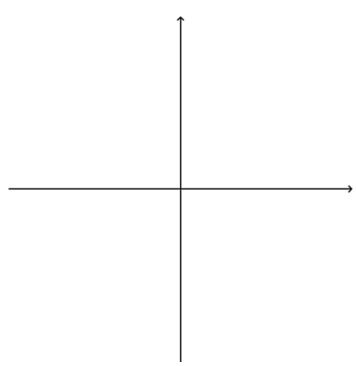
8. $f(x) = \frac{1}{x^2}$

Roots:

Vertical Asymptotes:

Horizontal Asymptotes:

Sign Line:



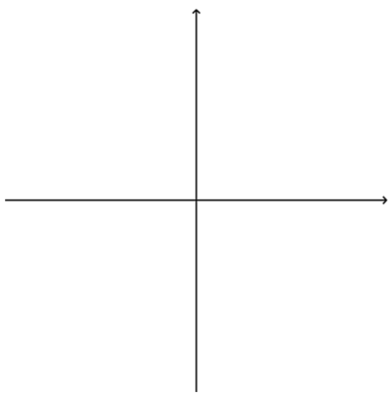
9. $f(x) = \frac{1}{x^2-9}$

Roots:

Vertical Asymptotes:

Horizontal Asymptotes:

Sign Line:



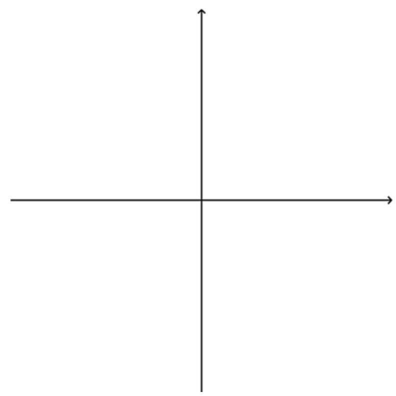
10. $f(x) = \frac{2}{(x-1)^2(x+2)}$

Roots:

Vertical Asymptotes:

Horizontal Asymptotes:

Sign Line:



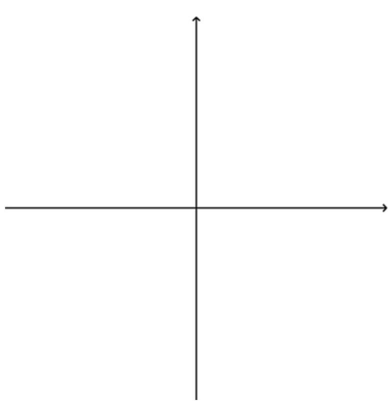
11. $f(x) = \frac{3x^2}{x^2-2x-24}$

Roots:

Vertical Asymptotes:

Horizontal Asymptotes:

Sign Line:



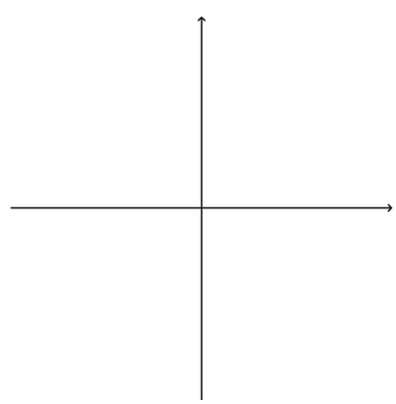
12. $f(x) = \frac{(x-1)(x+2)}{(x^3+4x^2+3x)}$

Roots:

Vertical Asymptotes:

Horizontal Asymptotes:

Sign Line:



13. Explain how you can determine if a rational function has roots.

14. Explain how you find the vertical asymptotes of a rational function.

4.9 Graphing Rational Functions
A Practice Understanding Task

For each function, determine the domain, intercepts, asymptotes, and complete a sign line. Use this information to sketch the graph.

1. $f(x) = \frac{x^2+1}{x(x-2)}$

   Domain:

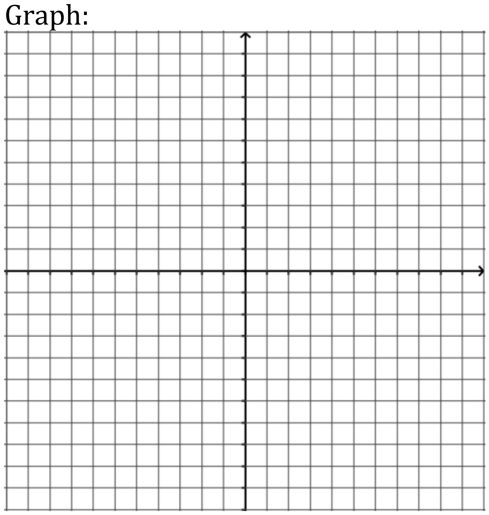
   Intercepts:

   Vertical Asymptote(s):

   Horizontal Asymptote:

   Sign Line:

   Graph:



2. $f(x) = \frac{2x}{(x-1)^2(x+2)}$

   Domain:

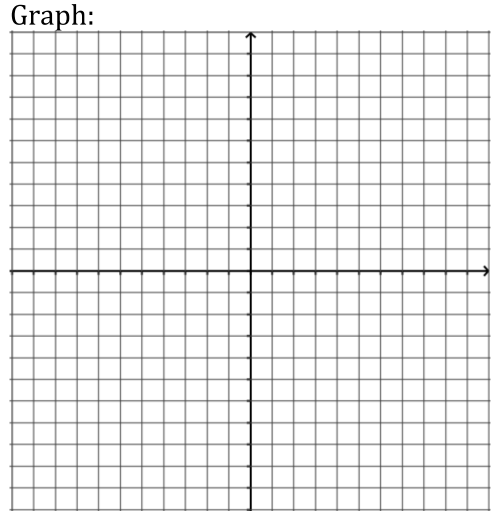
   Intercepts:

   Vertical Asymptote(s):

   Horizontal Asymptote:

   Sign Line:

   Graph:



3. $f(x) = \frac{2x^2}{x+1}$

Domain:

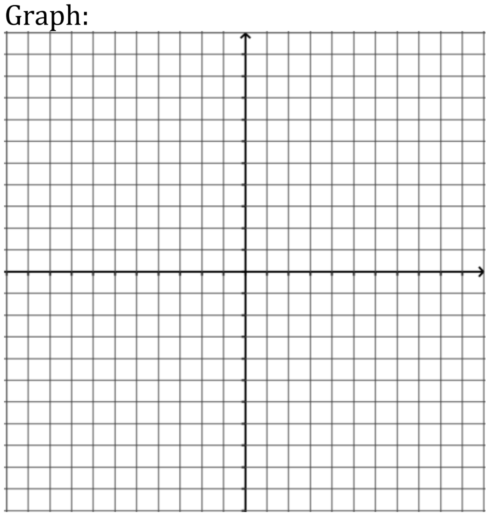
Intercepts:

Vertical Asymptote(s):

Horizontal Asymptote:

Sign Line:

Graph:



4. $f(x) = \frac{(x-1)^2}{x^3+4x}$

Domain:

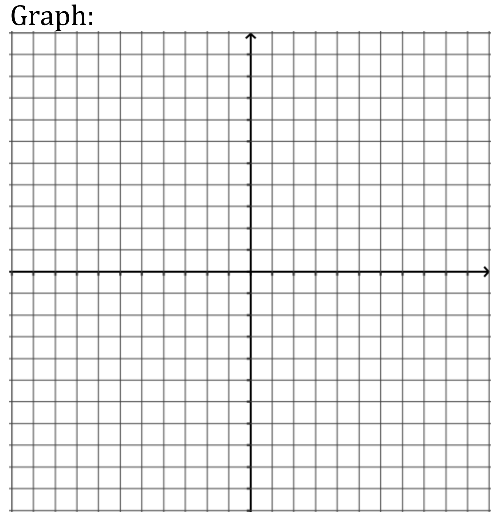
Intercepts:

Vertical Asymptote(s):

Horizontal Asymptote:

Sign Line:

Graph:



5. $f(x) = \frac{3x^2}{x^2-9}$

Domain:

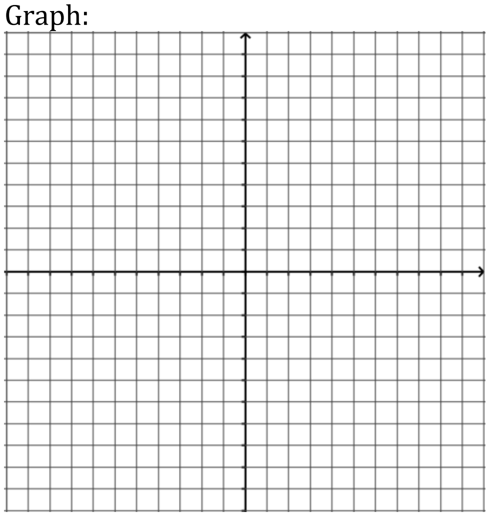
Intercepts:

Vertical Asymptote(s):

Horizontal Asymptote:

Sign Line:

Graph:



6. $f(x) = \frac{x}{x^2+1}$

Domain:

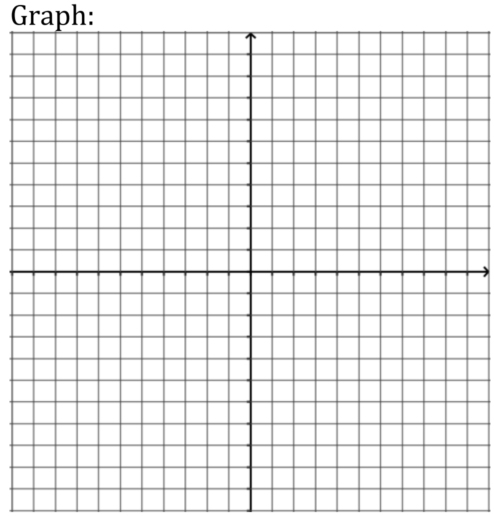
Intercepts:

Vertical Asymptote(s):

Horizontal Asymptote:

Sign Line:

Graph:



7. $f(x) = \frac{(2x-1)(x+2)}{(x+3)(x-1)}$

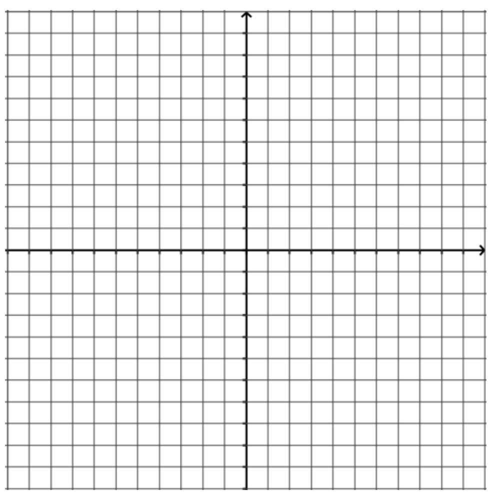
Domain:

Intercepts:

Vertical Asymptote(s):

Horizontal Asymptote:

Sign Line:



8 $f(x) = \frac{x^4-2x^2-3}{x^2}$

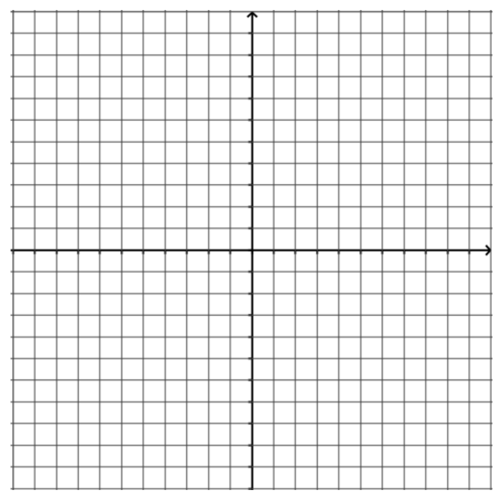
Domain:

Intercepts:

Vertical Asymptote(s):

Horizontal Asymptote:

Sign Line:



9. $f(x) = \frac{2x}{(x-1)^2}$

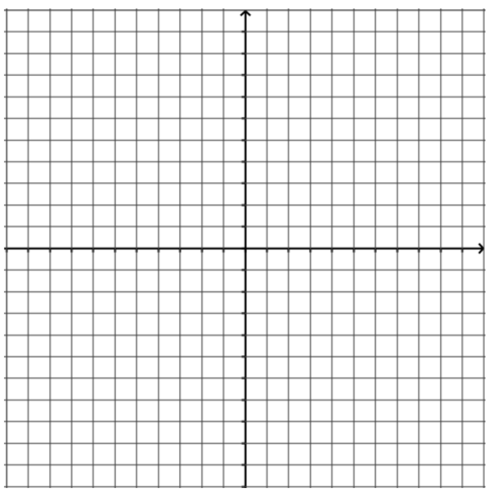
Domain:

Intercepts:

Vertical Asymptote(s):

Horizontal Asymptote:

Sign Line:



10. What features of rational functions make each graph unique?

Integrated Math 3
Module 4
Rational Functions
Ready, Set, Go! Homework

Adapted from

The Mathematics Vision Project:
Scott Hendrickson, Joleigh Honey, Barbara Kuehl,
Travis Lemon, Janet Sutorius

Name: **Rational Functions | 4.1**

Ready, Set, Go!

Ready

Topic: Polynomial division

Use division to determine if the given linear term is a factor of the polynomial.

1. $x^3 + 6x^2 + 8x; x + 2$

2. $b^2 - 9b - 10; b + 1$

3. $x^2 + 9x + 14; x + 7$

4. $2x^2 - 5x - 1; x - 3$

5. $x^3 + 10x^2 + 13x + 36; x + 9$

6. $m^3 + m^2 - 36m + 42; m + 5$

Set

Topic: Simplifying rational expressions & operations on rational expressions

Answer the following questions.

7. Define *rational expression* and give three examples.

8. Circle the expressions below that are rational. Explain why the non-circled terms are irrational.

$\frac{x^2+5}{4x-1}$ $\frac{3}{x^2-16}$ $\frac{\sin x}{2x+4}$ $\frac{x^2+6x+5}{x^3+7x-9}$ $\sqrt{x+3}$ $\frac{x^2+6x+5}{5}$

9. Angela simplified the following rational expressions. Circle the one(s) she answered correctly. Then, identify and given the correct answers in the other two problems.

a. $\frac{5x}{(x-3)}+\frac{2}{(x-1)}$

$\frac{5x(x-1)}{(x-3)(x-1)}+\frac{2(x-3)}{(x-3)(x-1)}$

$\frac{5x^2-x+2x-6}{(x-3)(x-1)}$

$\frac{5x^2+x-6}{(x-3)(x-1)}$

b. $\frac{x}{(x+3)}-\frac{4(x+3)}{(x-1)}$

$\frac{x}{1}-\frac{4}{(x-1)}$

$\frac{x(x-1)}{(x-1)}-\frac{4}{(x-1)}$

$\frac{x^2-x-4}{(x-1)}$

c. $\frac{(x+1)(x-2)}{(x+2)}\cdot\frac{(x+5)}{(x-2)(x+2)}$

$\frac{(x+1)(x-2)(x+5)}{(x+2)(x-2)(x+2)}$

$\frac{(x+1)(x+5)}{(x+2)(x+2)}$

$\frac{x^2+6x+5}{x^2+4x+4}$

Simplify each expression. Leave answers in factored form.

10. $\frac{2x+6}{(x+1)}-\frac{4}{(x+1)}$

11. $\frac{2x}{x+2}+\frac{x-1}{x-5}$

12. $\frac{x^2+6x+8}{x^2-5x+4}\cdot\frac{x^2+3x-4}{x^2+4x+4}$

13. $\frac{x^2+6x+8}{x^2-5x+4}\div\frac{x^2+3x-4}{x^2+4x+4}$

Go

Topic: Solving radical equations

Find ALL solutions to the following equations. Watch out for extraneous solutions (answers that make the original equation false).

14. $\sqrt{x+5} = 4$

15. $\sqrt{2x+15} = x+6$

16. $2\sqrt{x-1} = 2$

17. $\sqrt{3x+19} = x-3$

18. a. What would cause an extraneous solution when solving a **radical equation**?

    b. What would cause an extraneous solution when solving a **rational equation**?

Name: **Rational Functions | 4.2**

Ready, Set, Go!

Ready

Topic: Identifying values that make the expressions undefined

Identify the value(s) of x that make each expression undefined (hint: make the denominator equal to 0 or make the radicand negative).

1. $\frac{2}{x-6}$

2. $\sqrt{x+4}$

3. $\frac{x+5}{x+2}$

4. $\sqrt{2x-1}$

5. $\frac{1}{(x+3)(x-9)}$

6. $\frac{4x}{x^2+7x+10}$

Set

Topic: Solving rational equations

Find ALL solutions to the following equations. Watch out for extraneous solutions (answers that make the original equation false).

7. $\frac{4}{x+2}+\frac{2x}{x-1}=\frac{2x+1}{x+2}$

8. $\frac{x^2+9x+18}{x+3}=2$

9. $\frac{3}{x+1}+\frac{4}{x+2}=5$

10. $\frac{x+3}{x+2}=1-\frac{x+1}{x+2}$

11. $\frac{x}{x+1}=\frac{5}{2x-2}-\frac{1}{2}$

12. $\frac{4x}{x+1}+\frac{x+5}{x+1}=3$

Go

Topic: Operations on rational expressions

Simplify each expression completely.

13. $\frac{7}{9x^2}+\frac{x}{3x^2+3x}$

14. $\frac{7x}{2x-1}\div\frac{x^2-6x}{x^2-11x+30}$

15. $\frac{x+2}{2x-2}-\frac{-2x-1}{x^2-4x+3}$

16. $\frac{x^2+3x-4}{x^2+4x+4}\cdot\frac{2x^2+4x}{x^2-4x+3}$

17. $\frac{x^2-4x-5}{x+5} \div (x^2+6x+5)$

18. $\frac{x+3}{x^2-2x-8} - \frac{x-5}{x^2-12x+32}$

Ready, Set, Go!

Ready
Topic: Domain and range

State the domain and range of the following functions.

1. $f(x) = 2(x+6)^2 - 1$

   Domain:

   Range:

2. $f(x) = -3|x-4| + 8$

   Domain:

   Range:

3. $f(x) = 2x + 9$

   Domain:

   Range:

4. $f(x) = \sqrt{x-3} + 5$

   Domain:

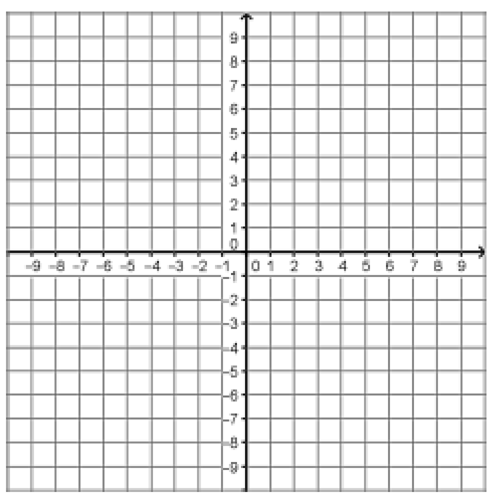
   Range:

Set
Topic: Features of rational functions

For each function, fill in the table of values and then graph the function. Then list the features of the function (domain/range, continuous/not continuous, intercepts, etc.).

5. $f(x) = \frac{1}{x-1}$

x	$f(x)$
-1	
0	
$\frac{1}{2}$	
1	
$1\frac{1}{2}$	
2	
3	



List of Features:
Domain:

Range:

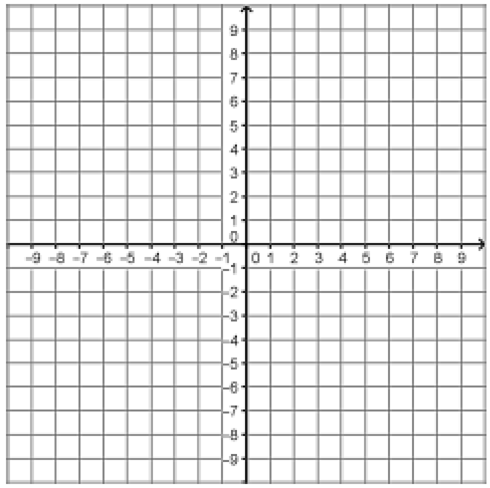
Continuous:

Intercepts:

End behavior:

6. $f(x) = \frac{1}{x+2}$

x	$f(x)$
-4	
-3	
$-2\frac{1}{2}$	
-2	
$-1\frac{1}{2}$	
-1	
0	



List of Features:
Domain:

Range:

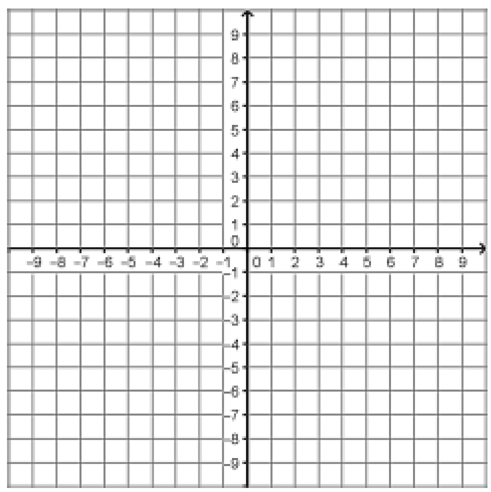
Continuous:

Intercepts:

End behavior:

7. $f(x) = \frac{3}{x}$

x	$f(x)$
-2	
-1	
$-\frac{1}{2}$	
0	
$\frac{1}{2}$	
1	
2	



List of Features:
Domain:

Range:

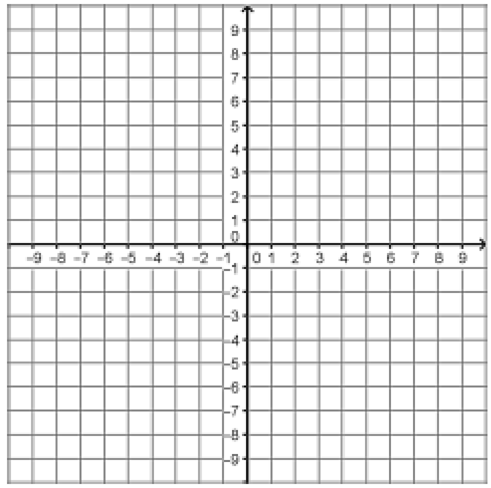
Continuous:

Intercepts:

End behavior:

8. $f(x) = -\frac{1}{x}$

x	$f(x)$
-2	
-1	
$-\frac{1}{2}$	
0	
$\frac{1}{2}$	
1	
2	



List of Features:
Domain:

Range:

Continuous:

Intercepts:

End behavior:

9. What happens to $f(x)$ as the x-values get closer to 0 in the graph in question 8?

Go

Topic: Finding solutions for polynomial functions.

Find all of the solutions for the given polynomials.

10. $x^2 + 4x + 3 = 15$

11. $x^3 + 5x^2 + 6x = 0$

12. $2x^2 + x + 2 = 5$

13. Given $x^3 - 7x + 6 = 0$ and $x = 1$ is a solution, find the remaining two solutions

Name: **Rational Functions** **4.4**

Ready, Set, Go!

Ready
Topic: Describing transformations

Describe how the parent function was transformed to obtain each of the following functions.

1. $f(x) = 2(x-3)^2 - 7$

2. $f(x) = \sqrt{x+6} + 2$

3. $f(x) = -\frac{1}{2}|x+4| - 5$

4. $f(x) = -\frac{5}{2}(x-1)^2 + 2$

Set
Topic: Features and families of functions

Find the roots and domain of each function. State the equations of any vertical asymptotes, if they exist.

5. $f(x) = (x+5)(x-2)(x-7)$

   Zeros:
   Domain:
   Asymptotes:

6. $g(x) = x^2 + 7x + 6$

   Zeros:
   Domain:
   Asymptotes:

7. $k(x) = \frac{(x+1)(x+6)}{(x+2)}$

   Zeros:
   Domain:
   Asymptotes:

8. $h(x) = \frac{-x+2}{(x+1)(x+5)}$

   Zeros:
   Domain:
   Asymptotes:

9. $q(x) = \sqrt{x+2} - 1$

Zeros:
Domain:
Asymptotes:

10. $f(x) = 5x^2 - 17x + 14$

Zeros:
Domain:
Asymptotes:

11. $p(x) = \frac{x-4}{x-3}$

Zeros:
Domain:
Asymptotes:

12. $m(x) = \frac{2x+5}{(x-8)(x+4)}$

Zeros:
Domain:
Asymptotes:

The rational parent function, $p(x)$, is defined below. Sketch a graph of the parent function. Then sketch the graphs of "parents transformed." Write each $t(x)$ and $m(x)$ function in terms of *x*.

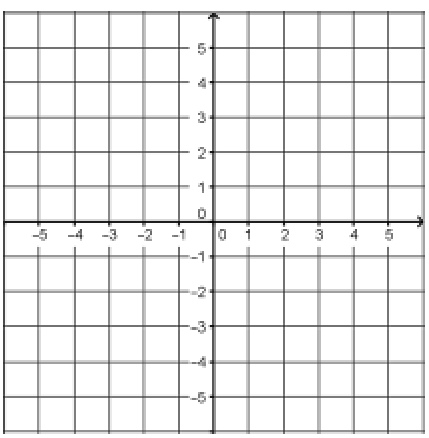
13. $p(x) = \frac{1}{x}$

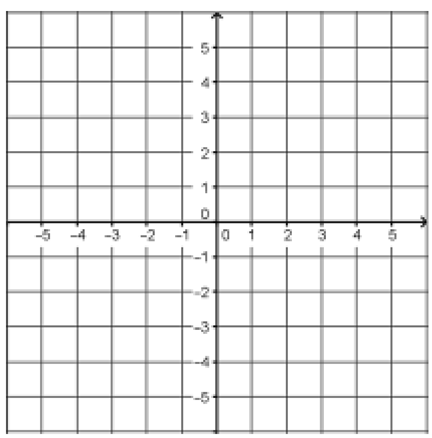
$t(x) = p(x) + 3$

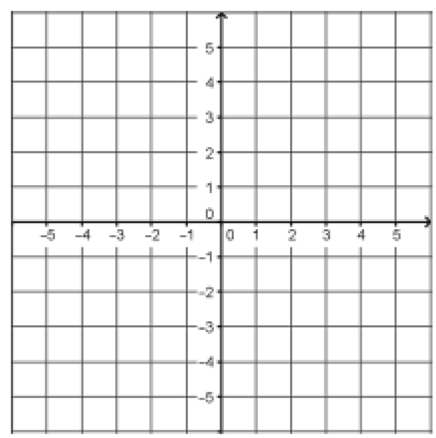
$t(x) =$

$m(x) = p(x-2)$

$m(x) =$







Go

Topic: Operations with rational expressions

Simplify each expression completely.

14. $\frac{7x+7}{x^2+18x+80} \cdot \frac{x+8}{7x+7}$

15. $\frac{3}{x-5} + \frac{6}{3x-8}$

16. $\frac{x^2+8x+16}{x^2-6x+9} \div \frac{2x+8}{3x-9}$

17. $\frac{3}{x+1} + \frac{4}{2x-6} - \frac{x^2-5}{x^2-2x-3}$

Topic: Solving rational equations

Solve each equation.

18. $\frac{5}{x^3+5x^2} = \frac{4}{x+5} + \frac{1}{x^2}$

19. $\frac{x+5}{x^2+x} = \frac{1}{x^2+x} - \frac{x-6}{x+1}$

20. $\frac{3}{x^2+5x+6} - \frac{x-6}{x^2+5x+6} = \frac{1}{x+3}$

21. $\frac{x}{x+4} = 3 - \frac{4}{x+4}$

Name: **Rational Functions | 4.5**

Ready, Set, Go!

Ready

Topic: Distinguishing between proper and improper rational functions.

Determine if each of the following is a proper or an improper rational function. (Hint: look at the degree of the polynomials.)

1. $f(x) = \frac{x^3+3x^2+7}{7x^2-2x+1}$

2. $f(x) = x^3 - 5x^2 - 4$

3. $f(x) = \frac{3x^2-2x+7}{x^5-5}$

4. $f(x) = \frac{x^3+4x^2+2x}{10x+7}$

5. $f(x) = \frac{5x^2-4x+4}{7x^5-2x+3}$

Set

Topic: Features of rational functions.

Find the *x*-intercept(s), *y*-intercept, and any vertical asymptotes of the following functions.

6. $f(x) = \frac{(x-1)(x+4)}{(x-5)(x+1)(x+2)}$

   x-intercept(s):

   y-intercept:

   vertical asymptotes:

7. $g(x) = \frac{(x^2-1)}{(x-3)(x+2)^2}$

   x-intercept(s):

   y-intercept:

   vertical asymptotes:

Topic: Improper vs. proper rational expressions

Determine if each rational expression is proper or improper. If improper, divide the polynomials to rewrite the rational expressions such that $\frac{a(x)}{b(x)} = q(x) + \frac{r(x)}{b(x)}$ where $q(x)$ represents the quotient and $r(x)$ represents the remainder.

8. $\frac{2x^3-7x^2+6}{x-3}$

9. $\frac{(x+1)}{(x-2)(x+2)}$

10. $\frac{x^3-3x^2+5x-1}{x^2-4x+4}$

11. $\frac{x^3-5x+2}{x-10}$

Go

Topic: Simplifying rational expressions

Simplify each of the rational expressions by canceling common factors. Leave answers in factored form, where possible.

12. $\frac{x^2+7x+12}{x^2+5x+6}$

13. $\frac{2x^2-8}{x^2-4x+4}$

14. $\frac{x+4}{x^2+6x+8}$

15. $\frac{n+3}{4} \cdot \frac{2(n-6)}{n+3}$

16. $\frac{x^2-4}{x+3} \div \frac{x+2}{x^2-9}$

Topic: Solving rational equations.

Solve each rational equation. Be sure to check for extraneous solutions.

17. $\frac{1}{x-2} = \frac{3}{x+2} - \frac{6x}{x^2-4}$

18. $\frac{1}{x-6} + \frac{x}{x-2} = \frac{4}{x^2-8x+12}$

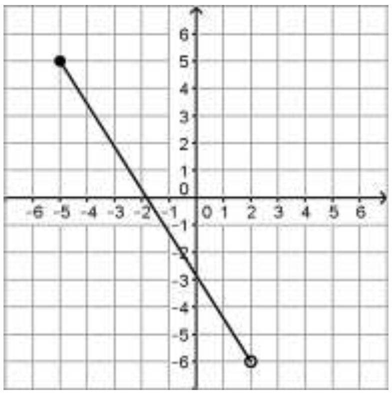
Ready, Set, Go!

Ready

Topic: Domain and range

Based on the graph given in each problem below, identify the domain and range of each function.

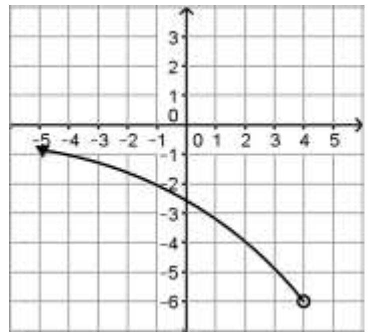
1.



Domain:

Range:

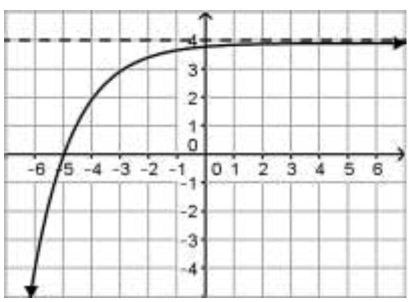
2.



Domain:

Range:

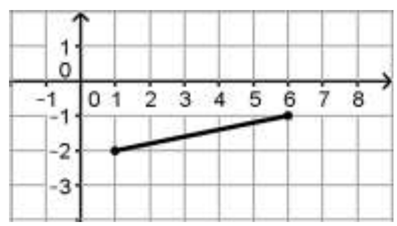
3.



Domain:

Range:

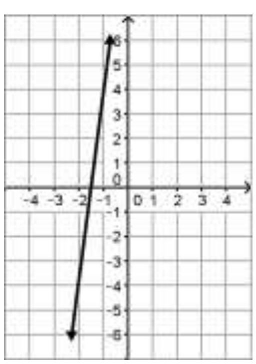
4.



Domain:

Range:

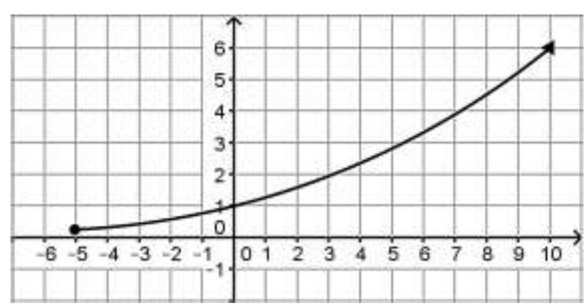
5.



Domain:

Range:

6.



Domain:

Range:

Set

Topic: Determine end behavior for rational functions

For each of the given functions, use a graphing utility to describe the end behavior as the *x*-values approach $+\infty$ and also $-\infty$.

7. $f(x) = \frac{(x-1)(x+2)}{x}$

   As $x \to \infty, f(x) \to$

   As $x \to -\infty, f(x) \to$

8. $g(x) = \frac{2x+3}{x+1}$

   As $x \to \infty, f(x) \to$

   As $x \to -\infty, f(x) \to$

9. $h(x) = \frac{(x-1)(x+2)}{2x}$

   As $x \to \infty, f(x) \to$

   As $x \to -\infty, f(x) \to$

10. $t(x) = \frac{x^3+2x^2+x}{x+2}$

    As $x \to \infty, f(x) \to$

    As $x \to -\infty, f(x) \to$

11. $p(x) = \frac{0.001(x^4+x^2+4)}{x}$

    As $x \to \infty, f(x) \to$

    As $x \to -\infty, f(x) \to$

12. $g(x) = \frac{2x+1}{x^2}$

    As $x \to \infty, f(x) \to$

    As $x \to -\infty, f(x) \to$

Topic: Finding vertical and horizontal asymptotes of rational functions.

Find the vertical, any horizontal asymptotes, and *x*-intercepts for the functions below.

13. $f(x) = \frac{1}{x^2-9}$

    Vertical:

    Horizontal:

    x-intercepts:

14. $f(x) = \frac{3}{x^3+2x^2-3x}$

    Vertical:

    Horizontal:

    x-intercepts:

15. $f(x) = \frac{1}{x^2}$

    Vertical:

    Horizontal:

    x-intercepts:

16. $f(x) = \frac{3x-1}{x+2}$

    Vertical:

    Horizontal:

    x-intercepts:

17. $f(x) = \frac{3x^2+4}{2x^2}$

Vertical:

Horizontal:

x-intercepts:

18. $f(x) = \frac{(2x-1)(2x+1)}{x+4}$

Vertical:

Horizontal:

x-intercepts:

Go

Topic: Rational equations.

Solve each rational equation.

19. $\frac{x-6}{2x^2+2x-4} + \frac{x}{2x-2} = \frac{1}{2}$

20. $1 = \frac{x-2}{x-1} + \frac{3}{x^2+3x-4}$

21. $0 = \frac{x^3+2x^2+x}{x+1}$

22. $0 = \frac{0.001(x^4+4x^2+4)}{x}$

Determine if each statement is true or false. If the statement is false, give a counterexample.

23. True or False: All polynomial functions are also rational functions.

24. True or False: All rational functions approach zero as $x \to \pm\infty$.

Name: **Rational Functions | 4.7**

Ready, Set, Go!

Ready

Topic: Vertical and horizontal asymptotes.

Determine the equations of any vertical and horizontal asymptotes of the following rational functions.

1. $f(x) = \frac{2}{x-4}$

   Vertical asymptotes:

   Horizontal asymptotes:

2. $f(x) = \frac{x-4}{x+3}$

   Vertical asymptotes:

   Horizontal asymptotes:

3. $f(x) = \frac{-1}{(x-3)(x+5)}$

   Vertical asymptotes:

   Horizontal asymptotes:

4. $f(x) = \frac{2x^2}{(x-6)(x+3)}$

   Vertical asymptotes:

   Horizontal asymptotes:

Set

Topic: Even and odd functions

5. Determine which of the following functions are even, odd or neither. Label them accordingly.

   a. $f(x) = x^2 - 3$

   b. $f(x) = \frac{1}{x}$

   c. $f(x) = x^2 + 4x + 4$

   d. $f(x) = |x|$

   e. $f(x) = -x^2 + 7$

   f. $f(x) = x^3 + x + 2$

   g. $f(x) = -5x^3 + 2x$

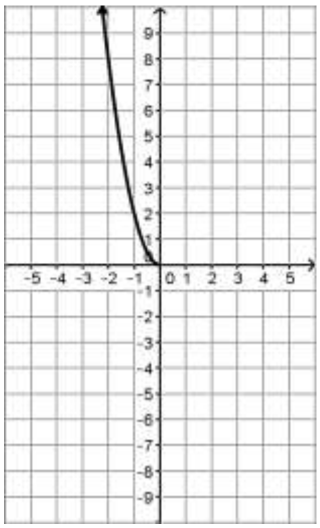
   h. $f(x) = -2(x+4)^2$

   i. $f(x) = |x+5|$

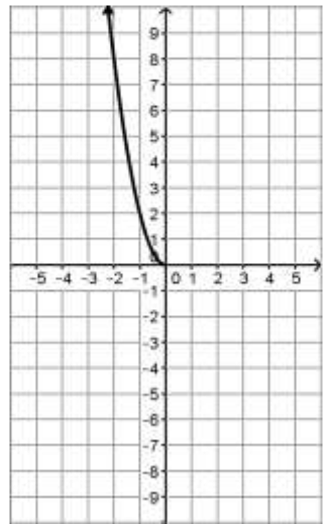
6. Use technology to graph each of the functions from number 5. What graphical characteristics go with an even function and what characteristics go with an odd function?

Given that the partial graphs below are even and odd functions, draw in the rest of the graph.

7. Even function



8. Odd Function



Go

Topic: Simplifying rational expressions

Simplify each expression. Where possible, leave your answers in factored form.

9. $\frac{x^2-8x+12}{x^2+3x-10}$

10. $\frac{7x^2-21x}{x^2-2x-35} \div \frac{x^2}{x-7}$

11. $\frac{3x-8}{x^2+6x+9} + \frac{5x}{x^2-9} + \frac{2}{x-3}$

12. $\frac{8x-1}{x^2+x-6} - \frac{4}{x-2}$

Topic: Solving rational equations

Solve each equation.

13. $\frac{3}{x^2-4} = \frac{2}{x+2} + \frac{x}{x-2}$

14. $\frac{2x+8}{x+3} - \frac{2}{x+3} = x$

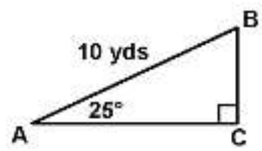
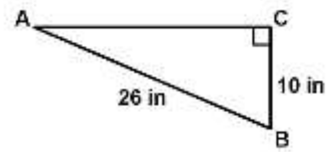
15. $\frac{3}{x+1} + \frac{x-2}{3} = \frac{13}{3x+3}$

Name: **Rational Functions | 4.8**

Ready, Set, Go!

Ready
Topic: Solving right triangles

Solve each right triangle by using sine, cosine, and/or tangent to find the missing side lengths and angle measures.

1. $\angle B =$ $\overline{AC} =$ $\overline{BC} =$ 	2. $\angle B =$ $\angle A =$ $\overline{AC} =$ 

Topic: Area of regular polygons

3. Break the polygon into congruent isosceles triangles. Use the isosceles triangles to find the measure of the central angle and the area of the regular polygon.

Measure of the central angle:

Area:

Set

Topic: Features of rational functions

Identify the features of the function, complete the sign line, and then sketch the function.

4. $f(x) = \frac{2x}{x-3}$

Domain:

Intercepts:

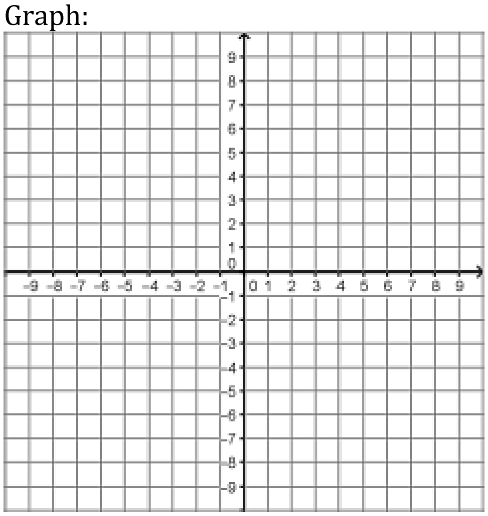
End Behavior:

Vertical Asymptote(s):

Horizontal Asymptote:

Sign Line:

Graph:



5. $g(x) = \frac{1}{(x-2)(x+2)}$

Domain:

Intercepts:

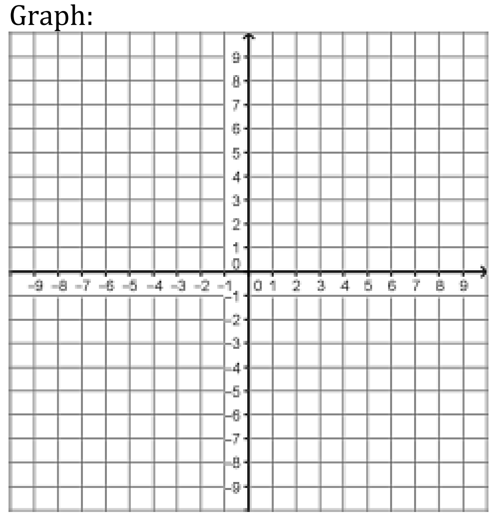
End Behavior:

Vertical Asymptote(s):

Horizontal Asymptote:

Sign Line:

Graph:



6. $h(x) = \frac{x-2}{x-1}$

Domain:

Intercepts:

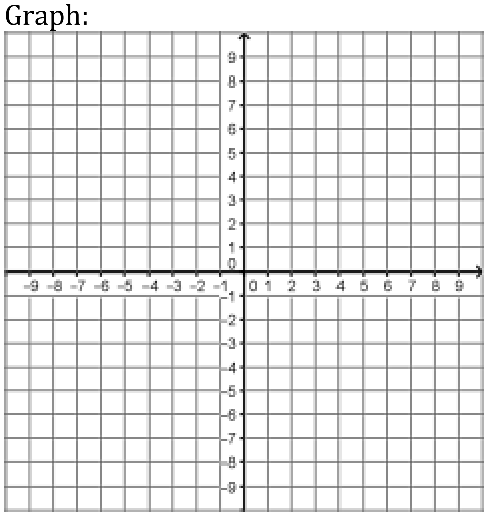
End Behavior:

Vertical Asymptote(s):

Horizontal Asymptote:

Sign Line:

Graph:



7. $w(x) = -\frac{1}{x^2}$

Domain:

Intercepts:

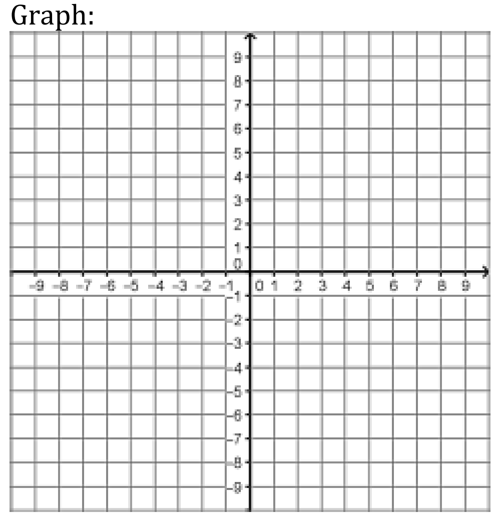
End Behavior:

Vertical Asymptote(s):

Horizontal Asymptote:

Sign Line:

Graph:



Go

Topic: Simplifying rational expressions

Rewrite each of the rational expressions in its simplest form. Where possible, leave answers in factored form.

8. $\frac{x^2+8x+12}{x^2+3x-18}$

9. $\frac{x^2-3x-40}{x^2-11x+24}$

10. $\frac{x^2+8x+12}{x^2+3x-18}+\frac{x^2-3x-40}{x^2-11x+24}$

11. $\frac{x^2+5x-36}{(x-4)}\cdot\frac{3(x+2)}{x-9}$

12. $\frac{x+3}{x^2-4}-\frac{x^2+4x+6}{x-2}$

13. $\frac{x^2-2x-3}{x+1}\div\frac{x-3}{5}$

Name: **Rational Functions | 4.9**

Ready, Set, Go!

Ready

Topic: Solving rational equations

Solve each rational equation. Be sure to check your solutions.

1. $\frac{2}{x+2} - \frac{1}{x} = \frac{1}{x}$

2. $\frac{5}{x-2} + \frac{7}{x+2} = \frac{10x-2}{x^2-4}$

3. $\frac{4}{x} = \frac{9}{x-2}$

4. $\frac{1}{4} + \frac{1}{x} = \frac{1}{6}$

5. $x + 2 + \frac{x}{x-2} = \frac{2}{x-2}$

6. $\frac{x}{x-2} + \frac{1}{x-4} = \frac{2}{x^2-6x+8}$

Set

Topic: Key features of rational functions

For each of the given functions list the key features of the function including the domain, the intercepts, end behavior, and asymptotes. Then complete the sign line and sketch a graph of each function.

7. $f(x) = \frac{3(x^2-1)}{x^2-4}$

Domain:

Intercepts:

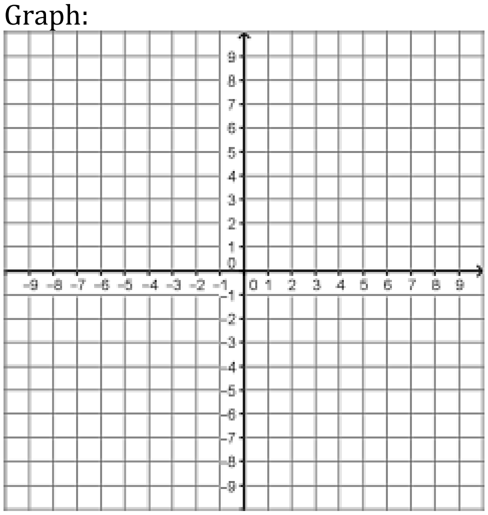
End Behavior:

Vertical Asymptote(s):

Horizontal Asymptote:

Sign Line:

Graph:



8. $r(x) = \frac{x+4}{x^2+5x-6}$

Domain:

Intercepts:

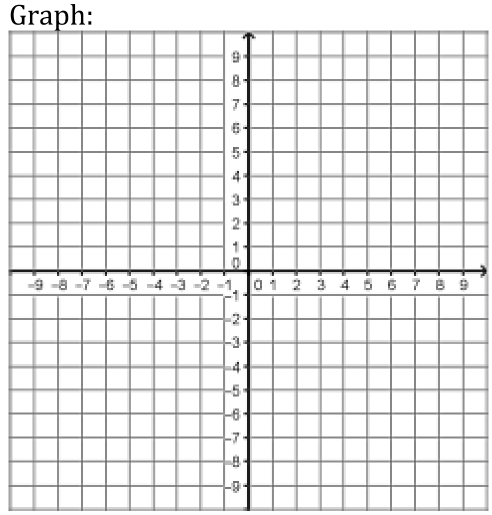
End Behavior:

Vertical Asymptote(s):

Horizontal Asymptote:

Sign Line:

Graph:



9. $q(x) = \frac{x}{x^3} + 3$

Domain:

Intercepts:

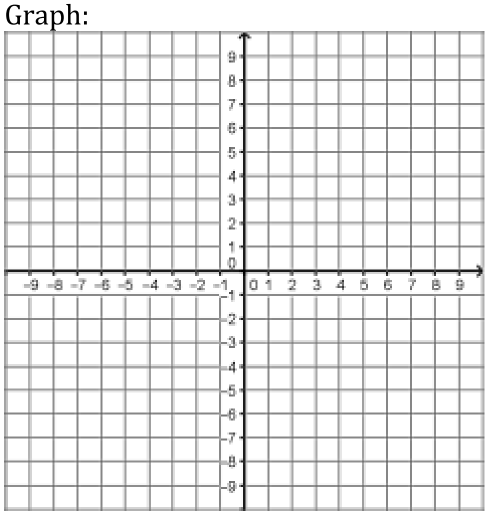
End Behavior:

Vertical Asymptote(s):

Horizontal Asymptote:

Sign Line:

Graph:



10. $m(x) = \frac{x-1}{2x+1}$

Domain:

Intercepts:

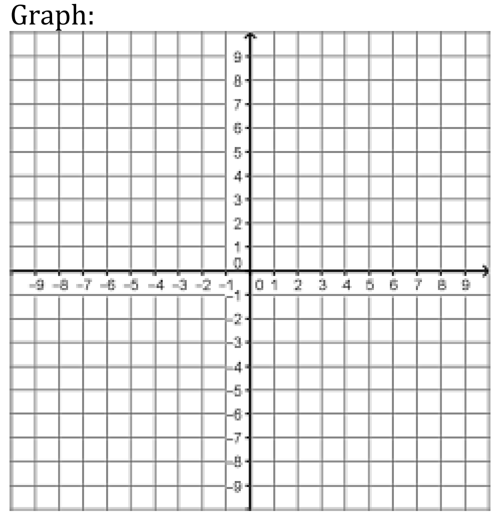
End Behavior:

Vertical Asymptote(s):

Horizontal Asymptote:

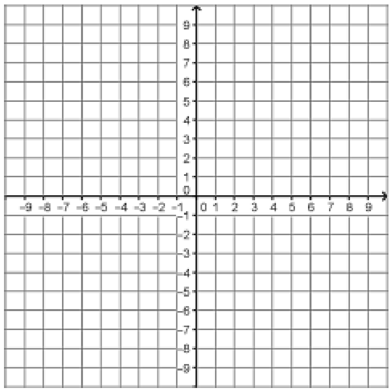
Sign Line:

Graph:

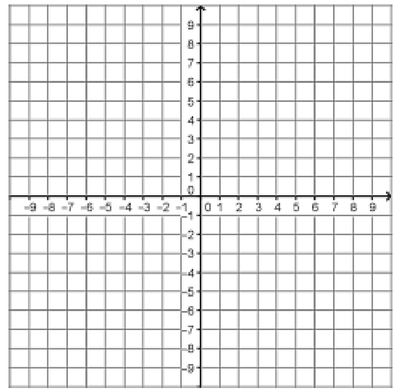


Sketch a graph and compare the given functions. Explain what you think accounts for the similarities and differences between the two functions.

11. a. $f(x) = \frac{1}{(x-2)(x+2)}$



b. $g(x) = \frac{x}{(x-2)(x+2)}$



Similarities:

Differences:

Why?

Go

Topic: Attributes of rational functions

12. How do you know if a function is even, odd, or neither?

13. How do you determine the end behavior of a rational function?

14. Is the end behavior for a rational function always the same?

15. What is the difference between a proper and an improper rational function?

16. What attributes do all proper rational functions have in common?

Topic: Operations on rational expressions

Perform the indicated operation. Be sure to simplify your solutions.

17. $\frac{2x}{x+1} - \frac{3x-2}{x}$

18. $\frac{x-4}{x-7} \cdot \frac{4x}{2x^2-32}$

19. $\frac{5x}{x^2+2x} \div \frac{30x^2}{x+2}$

20. $\frac{x-11}{x^2+6x-40} + \frac{5}{x-4}$

21. Use division to rewrite each improper rational expression:

a. $\frac{6x+5}{x+1}$

b. $\frac{4x+12}{x-2}$

Integrated Math 3
Module 5
Modeling with Geometry

Adapted from

The Mathematics Vision Project:
Scott Hendrickson, Joleigh Honey, Barbara Kuehl,
Travis Lemon, Janet Sutorius

Module 5 Overview

Prerequisite Concepts & Skills:
- Right triangle trigonometry
- Properties of quadrilaterals
- Area formulas for various shapes (quadrilaterals, triangles, regular polygons, circles)
- Volume formulas (prism, pyramid, cone)
- Altitude of a triangle
- Pythagorean Theorem
- Angle relationships within triangles and parallel lines

Summary of the Concepts & Skills in Module 5:
- Volume
- Figure Dissections
- Solids of Rotations
- Cross Sections
- Special Right Triangles
- Definitions of the reciprocal trigonometric functions
- Verifying and proving trigonometric identities

Content Standards and Standards of Mathematical Practice Covered:
- Content Standards: G.GMD.4, G.MG.1, G.MG.2, G.MG.3, G.SRT.10, G.SRT.11, F.IF.8, F.LE.4
- Standards of Mathematical Practice:
    1. Make sense of problems & persevere in solving them
    2. Reason abstractly & quantitatively
    3. Construct viable arguments & critique the reasoning of others
    4. Model with mathematics
    5. Attend to precision
    6. Use appropriate tools strategically
    7. Look for & make use of structure
    8. Look for & express regularity in repeated reasoning

Module 5 Vocabulary:
- Cross Section
- Cube
- Pyramid
- Prism
- Cylinder
- Vertex
- Edge
- Face

Concepts Used in the Next Module:
- Trigonometry
- Equation of a circle
- Special Right Triangles

Module 5 – Modeling with Geometry

Part 1:

5.1 *Visualizing two-dimensional cross sections of three-dimensional objects* **(G.GMD.4)**
Warm Up: Do You See What I See?
Classroom Task: Any Way You Slice It – A Develop Understanding Task
Ready, Set, Go Homework: Modeling with Geometry 5.1

5.2 *Visualizing solids of revolution* **(G.GMD.4)**
Warm Up: Can You Find It?
Classroom Task: Any Way You Spin It – A Develop Understanding Task
Ready, Set, Go Homework: Modeling with Geometry 5.2

5.3 *Approximating volumes of solids of revolution with cylinders and frustums* **(G.MG.1, G.GMD.4)**
Warm Up: Frustrating Figures
Classroom Task: Take Another Spin – A Solidify Understanding Task
Ready, Set, Go Homework: Modeling with Geometry 5.3

5.4 *Solving problems using geometric modeling* **(G.MG.1, G.MG.2, G.MG.3)**
Warm Up: Just a Slice
Classroom Task: Hard as Nails! – A Practice Understanding Task **(OPTIONAL)**
Ready, Set, Go Homework: Modeling with Geometry 5.4

Part 2:

5.5 *Examining the relationship of sides in special right triangles* **(G.SRT.11)**
Warm Up: A Special Area
Classroom Task: Special Rights – A Solidify Understanding Task
Ready, Set, Go Homework: Modeling with Geometry 5.5

5.6 *Defining cosecant, secant, and cotangent and identifying relationships between the six trigonometric functions* **(G.SRT.6, G.SRT.7, F.TF.8)**
Warm Up: Trigonometric Ratios
Classroom Task: More Relationships with Meaning– A Develop and Solidify Understanding Task
Ready, Set, Go Homework: Modeling with Geometry 5.6

5.7 *Verifying and proving trigonometric identities, including the Pythagorean identities* (**F.TF.8)**
Warm Up: Super Special Triangle
Classroom Task: Relationships with Meaning Parts (A Secret Identity) – A Solidify Understanding Task
Ready, Set, Go Homework: Similarity & Right Triangle Trigonometry 5.7

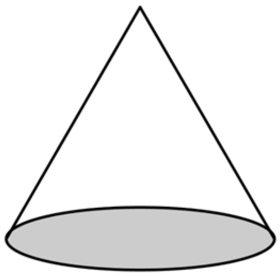
5.8 *Verifying trigonometric identities* (**F.TF.8)**
Warm Up: Simplifying Trigonometric Expressions
Classroom Task: Identity Verification– A Solidify Understanding Task
Ready, Set, Go Homework: Similarity & Right Triangle Trigonometry 5.8

5.9 *Review trigonometric identities through a card sort activity* **(F.TF.8)**
Classroom Task: It's a Match – A Practice Understanding Task
Ready, Set, Go Homework: Modeling with Geometry 5.9

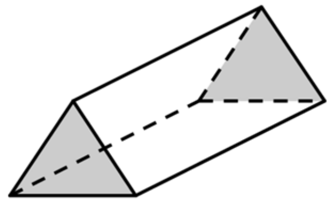
5.1 Warm Up
Do You See What I See?

Draw the front, side, and top views of the given solids:

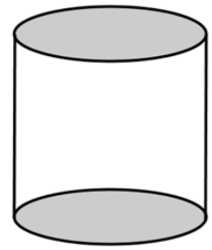
1.



2.



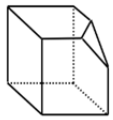
3.



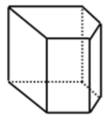
5.1 Any Way You Slice It
A Develop Understanding Task

Students in Mrs. Denton's class were given cubes made of clay and asked to slice off a corner of the cube with a piece of dental floss.

Jumal sliced his cube this way:



Jabari sliced his cube like this:



1. Which student, Jumal or Jabari, interpreted Mrs. Denton's instructions correctly? Why do you say so?

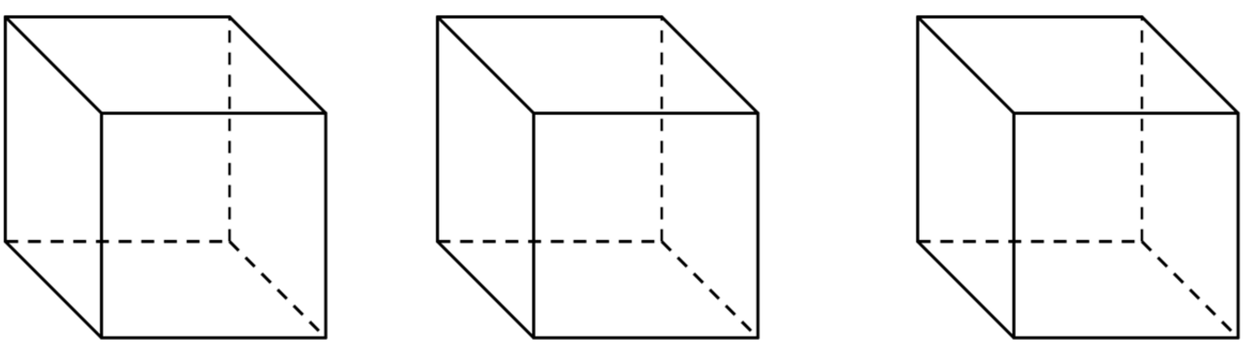
When describing three-dimensional objects such as cubes, prisms, or pyramids we use precise language such as ***vertex*, *edge*** or ***face*** to refer to the parts of the object in order to avoid the confusion that words like "corner" or "side" might create.

A ***cross section*** is the face formed when a three-dimensional object is sliced by a plane. It can also be thought of as the intersection of a plane and a solid.

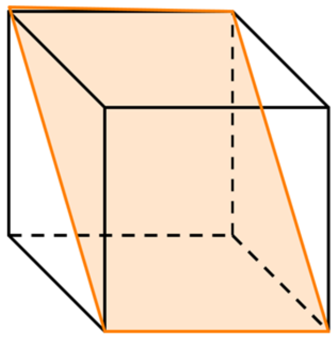
2. Draw and describe the cross section formed when Jumal sliced his cube.

3. Draw and describe the cross section formed when Jabari sliced his cube.

4. Draw some other possible cross sections that can be formed when a cube is sliced by a plane. It might be helpful to use a colored pen/pencil to shade the cross sections.



5. What type of quadrilateral is formed by the intersection of the plane that passes diagonally through opposite edges of a cube? Explain how you know which quadrilateral is formed by this cross section.



Jumal and Jabari visualized cross sections in many different ways:

- Cut a clay model of the solid with a piece of dental floss.
- Partially filled a clear glass or plastic model of the three-dimensional object with colored water and tilted it in various ways to see what shapes the surface of the water formed.
- Examined the two-dimensional shadow cast by the three-dimensional object as it was turned or rotated in the light.

Experiment with various ways of examining the cross sections of different three-dimensional shapes.

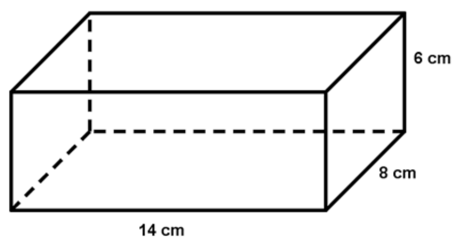
6. Partially fill a cylindrical jar with colored water, and tilt it in various ways. Draw the cross sections formed by the surface of the water in the jar.

7. Examine the shadow of a cube as it is positioned in various ways in front of a light source. Which of the following shadow-shapes can be formed? Which are impossible?

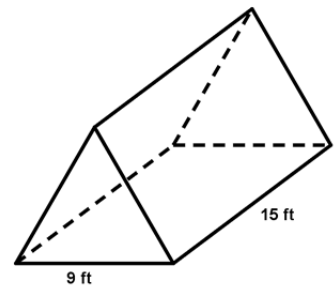
a square	a rhombus	a rectangle	a triangle
a pentagon	a hexagon	an octagon	a circle

Find the volume and surface area of each of the following solids.

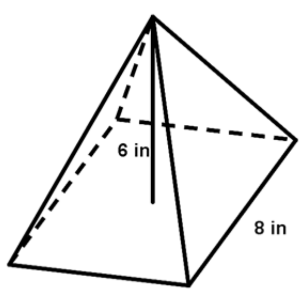
8. Rectangular Prism



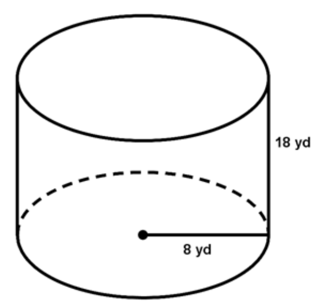
9. Equilateral Triangular Prism



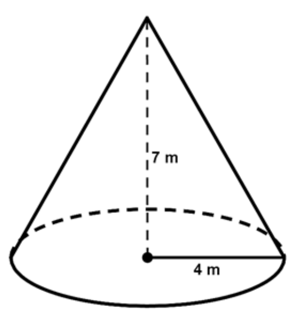
10. Square Pyramid



11. Cylinder

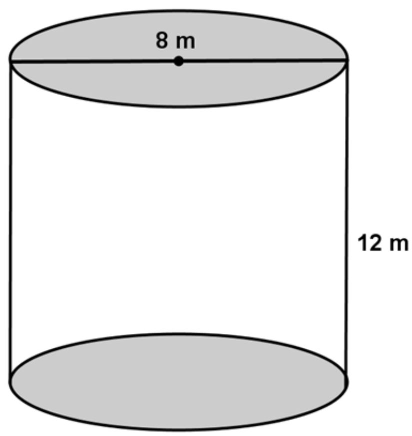


12. Cone



5.2 Warm Up
Can You Find It?

Find the surface area and volume of the cylinder below:

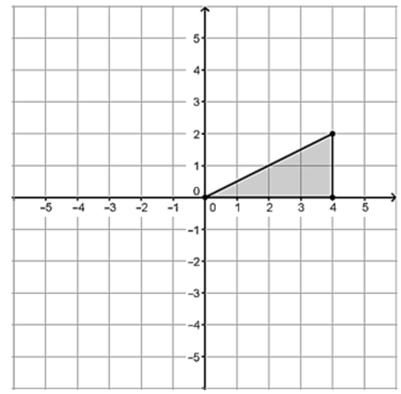


5.2 Any Way You Spin It
A Develop Understanding Task

Perhaps you have used a pottery wheel or a wood lathe. (A lathe is a machine that is used to shape a piece of wood by rotating it rapidly on its axis while a fixed tool is pressed against it. Table legs and wooden pedestals are carved on a wood lathe). You might have played with a spinning top or watched a figure skater spin so rapidly she looked like a solid blur. The clay bowl, the table leg, the rotating top and the spinning skater can be modeled as ***solids of revolution*** which is *a three dimensional object formed by spinning a two dimensional figure about an axis.*

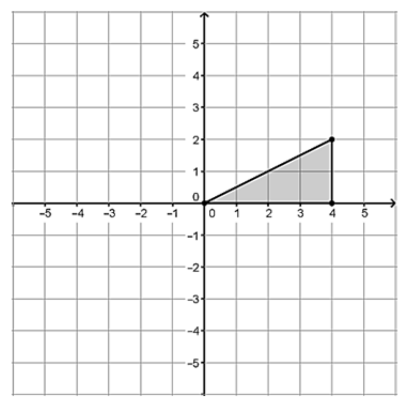
Suppose the right triangle shown below is rotating rapidly about the x-axis. Like the spinning skater, a solid image would be formed by the blur of the rotating triangle.

1. Draw and describe the solid of revolution formed by rotating this triangle about the x-axis.



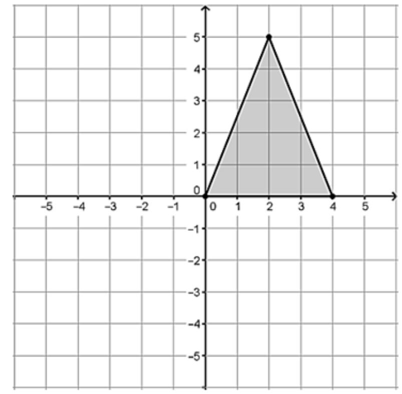
2. Find the volume of the solid formed.

3. What would this figure look like if the triangle rotates rapidly about the y-axis? Draw and describe the solid of revolution formed by rotating this triangle about the y-axis.



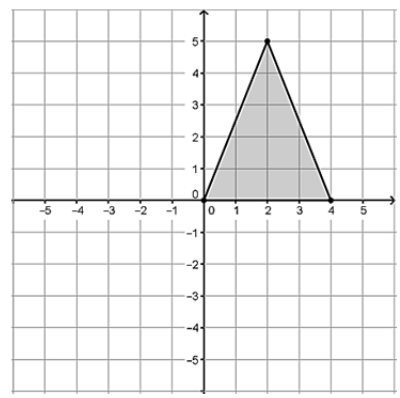
4. Find the volume of the solid formed.

5. Draw and describe the solid of revolution formed by rotating this triangle about the x-axis.



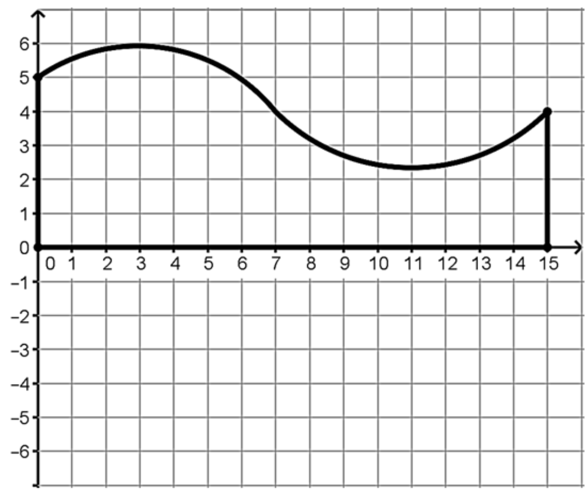
6. Find the volume of the solid formed.

7. What would this figure look like if the triangle rotates rapidly about the y-axis? Draw and describe the solid of revolution formed by rotating this triangle about the y-axis.



8. Find the volume of the solid formed.

9. What about the following two-dimensional figure? In the blank space below, draw and describe the solid of revolution formed by rotating this figure about the *x*-axis.



10. If the solid you drew in question 9 was cut by a plane that contains the axis of rotation (the *x*-axis), what would the cross section look like? Draw the cross section in the space below.

11. Draw some cross sections of the solid of revolution formed by the figure in question 9 if the planes cutting the solid are **perpendicular to the *x*-axis and parallel to the *y*-axis**. Draw the cross sections when the intersecting planes are located at $x = 5$, $x = 10$ and $x = 15$.

So, why are we interested in solids that don't really exist—after all, they are nothing more than a blur that forms an image of a solid in our imagination. **Solids of revolution are used to create mathematical models of real solids by describing the solid in terms of the two-dimensional shape that generates it.**

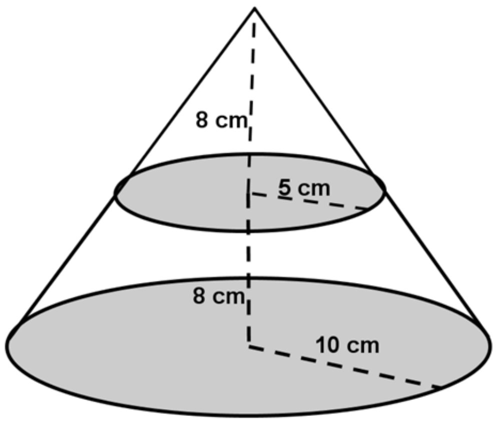
12. For each of the following solids

  a. Draw the axis of rotation on the object.

  b. Draw the two-dimensional shape that would be revolved about the *x*-axis that generates the solid.

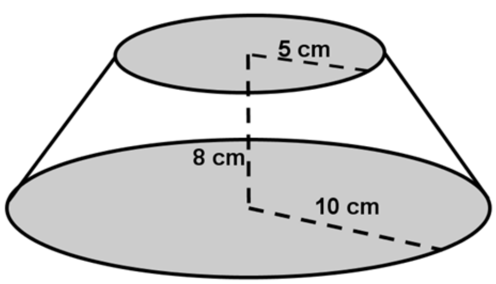
5.3 Warm Up
Frustrating Figures

1. Find the volume of the entire cone



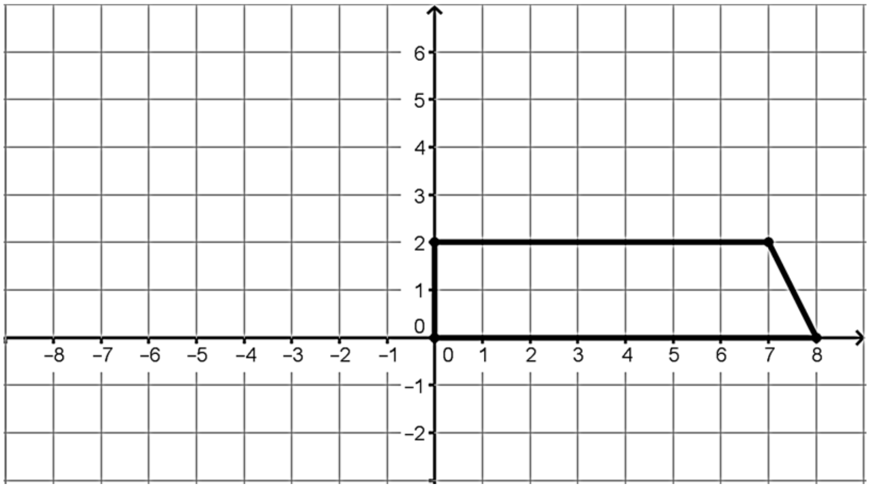
2. Find the volume of the smaller "top" cone.

3. Use your solutions to question 1 and 2 to find the volume of the figure below. (The figure is called a **frustum**.)



5.3 Take Another Spin
A Solidify Understanding Task

The trapezoid shown below is revolved about the y-axis to form a **frustum** (the portion of a cone or pyramid that remains after its upper part has been cut off).

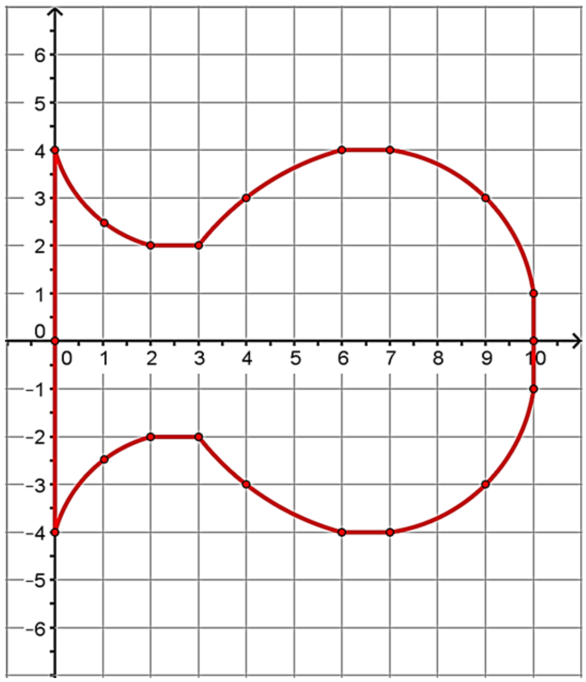


1. Draw a sketch of the three-dimensional object formed by rotating the trapezoid about the y-axis.

2. Find the volume of the object formed. Explain how you used the diagram to help you find the volume.

You have made use of the formulas for cylinders and cones in your work with solids of revolution. Sometimes a solid of revolution cannot be decomposed exactly into cylinders and cones. We can approximate the volume of solids of revolution whose cross sections include curved edges by replacing them with line segments.

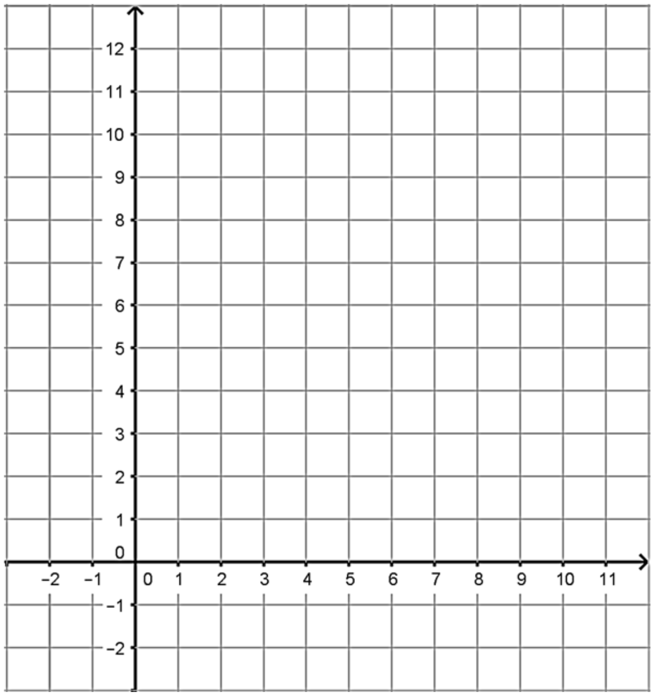
3. The following diagram shows the cross section of a flower vase. Approximate the volume of the vase by using line segments to approximate the curved edges. Show the line segments you used to approximate the figure on the diagram. Hint: Use six cross sections and avoid using spheres in your work.



4. Describe and carry out a strategy that will improve your approximation for the volume of the vase.

5.4 Warm Up
Just a Slice

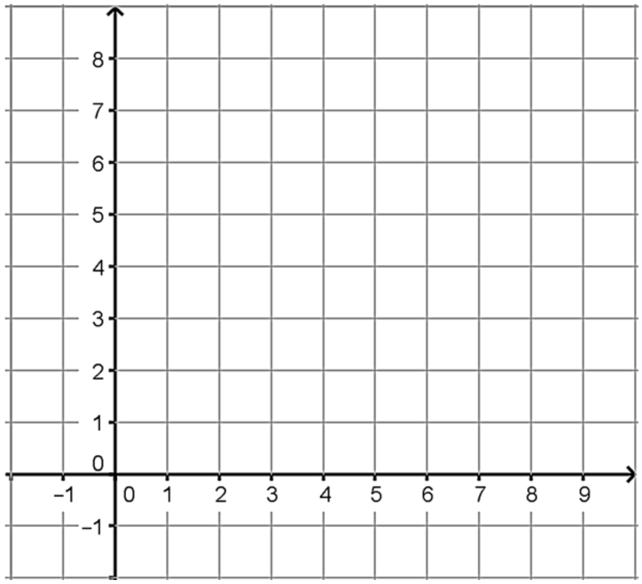
1. Shade the region below bounded by the line $y = -\frac{1}{2}x + 10$, the x-axis, $x = 2$, and $x = 8$.



2. Find the area of the shaded region.

3. If the region is rotated about the *x*-axis, a frustum is formed. Find the volume of the frustum.

4. Shade the region below bounded by the line $y = 2x - 6$, the x-axis, $x = 4$, and $x = 7$.



5. Find the area of the shaded region.

6. If the region is rotated about the x-axis, a frustum is formed. Find the volume of the frustum.

5.4 Hard as Nails
A Practice Understanding Task

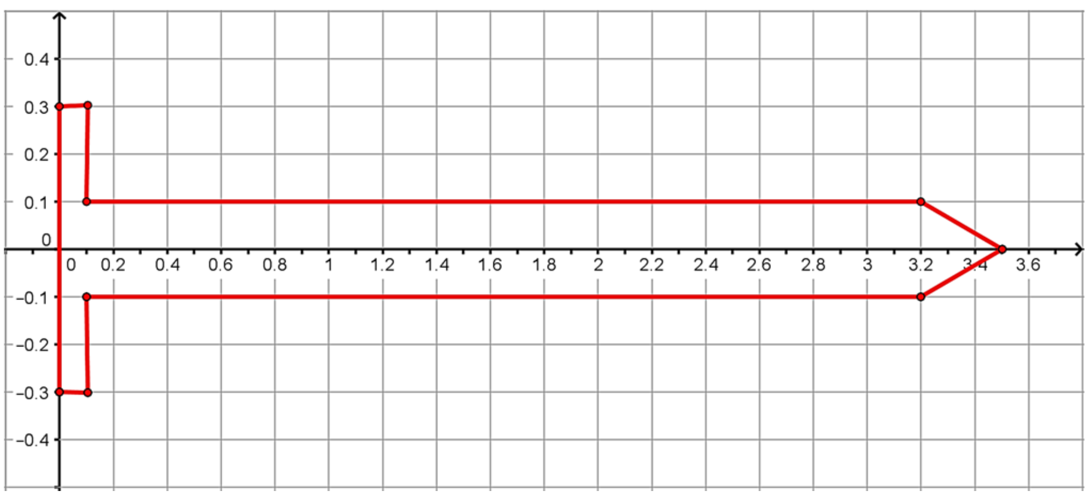
Tatiana is helping her father purchase supplies for a deck he is building in their backyard. Based on her measurements for the area of the deck, she has determined that they will need to purchase 24 decking planks. These planks will be attached to the framing joists with 16d nails.

(Tatiana thinks it is strange that these nails are referred to as "16 penny nails" and wonders where that way of naming nails comes from. After doing some research, Tatiana has found that in the late 1700's in England, the size of a nail was designated by the price of purchasing one hundred nails of that size. She doubts that her dad will be able to buy one hundred 16d nails for 16 pennies.)

Nails are sold by the pound at the local hardware store, so Tatiana needs to figure out how many pounds of 16d nails to tell her father to buy. She has gathered the following information:

- The deck requires 24 planks of wood
- Each plank requires 9 nails to attach it to the framing joists
- 16d nails are made of steel that has a density of approximately 4.57 oz/in^3
- There are 16 ounces in a pound

Tatiana has also found the following drawing of a cross section of a 16d nail. She knows she can use this drawing to help her find the volume of the nail, treating it as a solid of revolution. (Note: The scale on the *x*- and *y*-axis is in inches.)

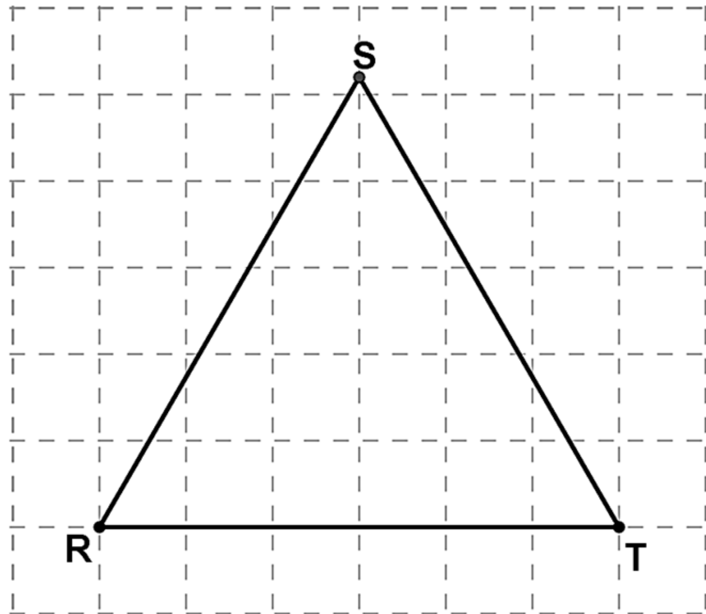


1. Devise a plan for finding the volume of the nail based on the given drawing. Describe your plan in words, and then show the computations that support your work.

2. Devise a plan for finding the number of pounds of 16d nails Tatiana's father should buy. Describe your plan in words, and then show the computations that support your work.

5.5 Warm Up
A Special Area

Investigation #1:



1. Given **equilateral** $\Delta \boldsymbol{RST}$ on the right, draw the altitude from S to $\overline{RT}$. Call the intersection point Q.

2. What happens to $\overline{RT}$ when the altitude is drawn in?

3. What are the measures of $\angle SRT$ and $\angle STR$? Label these measurements on the picture.

4. What are the measures of $\angle RSQ$ and $\angle TSQ$? Label these measurements on the picture.

5. In the two right triangles, which sides are the shortest? Explain how you determined this.

6. In the two right triangles, which sides are the longest? Explain how you determined this.

7. How do the smallest sides of the right triangles compare to the longest sides of the right triangles?

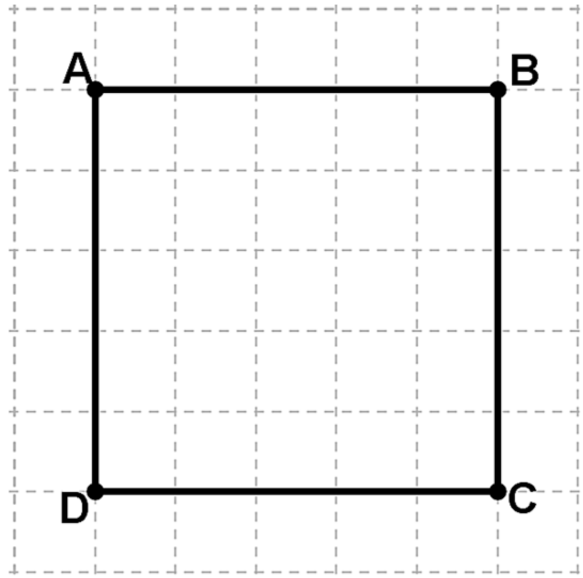
8. Label the smallest sides of the right triangles with the same variable.

9. Using the variable from #8, write an expression for the longest side of each right triangle.

10. Find an expression for the attitude, $\overline{SQ}$.

11. In mathematics, $30° - 60° - 90°$ triangles are one of the special right triangles that are often used. How are the sides of a $30° - 60° - 90°$ triangle related to each other? Why would this relationship be true for all $30° - 60° - 90°$ triangles?

Investigation #2:



1. Given **square ABCD** on the right. What is the definition of a square? Mark this definition on the diagram.

2. Draw the diagonal $\overline{AC}$.

3. What type of triangles are ΔACD and ΔACB? Explain how you determined this answer.

4. What are the measures of $\angle DAC$ and $\angle ACD$? Explain how you determined this answer. Label these measurements on the picture.

4. What are the measures of $\angle BAC$ and $\angle BCA$? Explain how you determined this answer. Label these measurements on the picture.

5. Label the legs of ΔACD with the same variable.

6. Using the variable from #5, write an expression for the hypotenuse of ΔACD.

7. In mathematics, $45° - 45° - 90°$ triangles are one of the special right triangles that are often used. How are the sides of a $45° - 45° - 90°$ triangle related to each other? Why would this relationship be true for all $45° - 45° - 90°$ triangles?

5.5 Special Rights
A Solidify Understanding Task

In previous courses you have studied the Pythagorean Theorem and right triangle trigonometry. Both of these mathematical tools are useful when trying to find missing sides of a right triangle.

1. What do you need to know about a right triangle in order to use the Pythagorean Theorem?

2. What do you need to know about a right triangle in order to use right triangle trigonometry?

While using the Pythagorean Theorem is fairly straight forward (you only have to keep track of the legs and hypotenuse of the triangle), right triangle trigonometry generally requires a calculator to look up values of different trigonometry ratios. There are some right triangles, however, for which knowing a side length and an angle measure is enough to calculate the value of the other sides without using trigonometry. These are known as ***special right triangles*** because their side lengths can be found by relating them to another geometric figure for which we know a great deal about its sides.

One type of special right triangle is a $45° - 45° - 90°$ triangle.

3. Draw a $45° - 45° - 90°$ triangle and assign a specific value to one of its sides. (For example, let one of the legs measure 5 cm, or choose to let the hypotenuse measure 8 inches. You will want to try both approaches to perfect your strategy.) Now that you have assigned a measurement to one of the sides of your triangle, find a way to calculate the exact measures of the other two sides. As part of your strategy, you may want to relate this triangle to another geometric figure that may be easier to think about.

4. Generalize your strategy for a $45° - 45° - 90°$ triangle by letting one side of the triangle measure x. Show how the exact measures of the other two sides can be represented in terms of x. Make sure to consider cases where x is the length of a leg, as well as the case where x is the length of the hypotenuse.

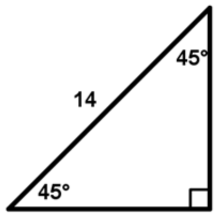
<u>Another type of special right triangle is a $30° - 60° - 90°$ triangle</u>.

5. Draw a $30° - 60° - 90°$ triangle and assign a specific value to one of its sides. Find a way to calculate the exact measures of the other two sides. As part of your strategy, you may want to relate this triangle to another geometric figure that may be easier to think about.

6. Generalize your strategy for $30° - 60° - 90°$ triangles by letting one side of the triangle measure *x*. Show how the exact measures of the other two sides can be represented in terms of *x*. Make sure to consider cases where *x* is the length of a leg, as well as the case where *x* is the length of the hypotenuse.

Find the missing sides of each special right triangle using the $45° - 45° - 90°$ or $30° - 60° - 90°$ triangle rules. Leave answers with simplified radicals, where necessary.

7.



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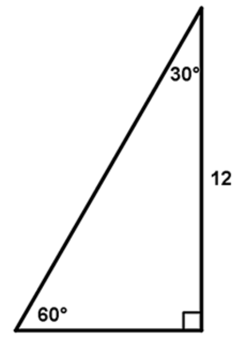

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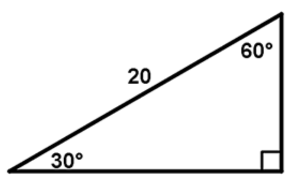

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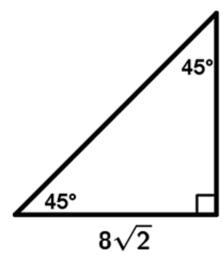

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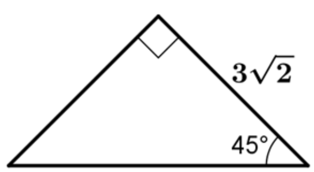

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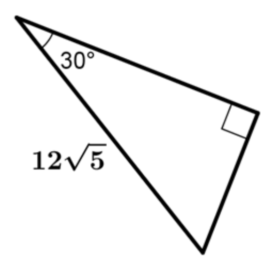

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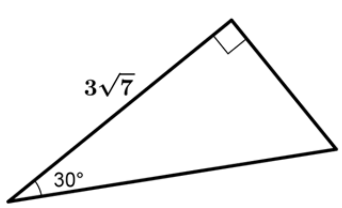

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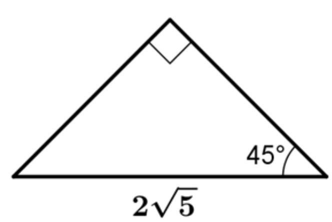
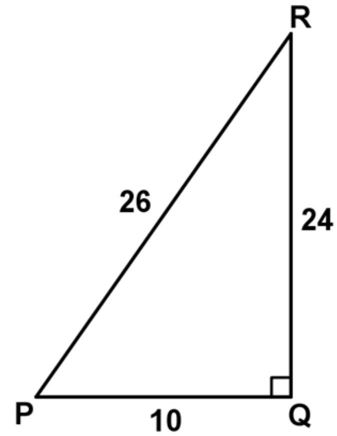


5.6 Warm Up
Trigonometric Ratios

Find the indicated trigonometric ratios for the right triangle shown.



1. $\sin P =$
2. $\cos P =$
3. $\tan P =$
4. $\sin R =$
5. $\cos R =$
6. $\tan R =$
7. Given $\sin\theta = \frac{5}{13}$, find the following (a drawing may be helpful):

   $\cos(90° - \theta) =$

   $\tan\theta =$

8. Given $\tan\theta = \frac{3}{4}$, find the following:

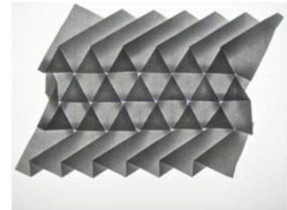
   $\sin\theta =$

   $\cos\theta =$

9. Previously, you looked at 3 ratios of sides of a right triangle when developing definitions of sine, cosine, and tangent $\left(\frac{opposite}{hypotenuse}, \frac{adjacent}{hypotenuse}, \text{and } \frac{opposite}{adjacent}\right)$.

   Are there any other ratios of sides possible for a right triangle? If yes, list the ratios. If no, explain why not.

5.6 More Relationships with Meaning
A Develop and Solidify Understanding Task



1. Use the information from the given triangle to write the following trigonometric ratios:

$\csc A = \frac{hypotenuse}{opposite} =$

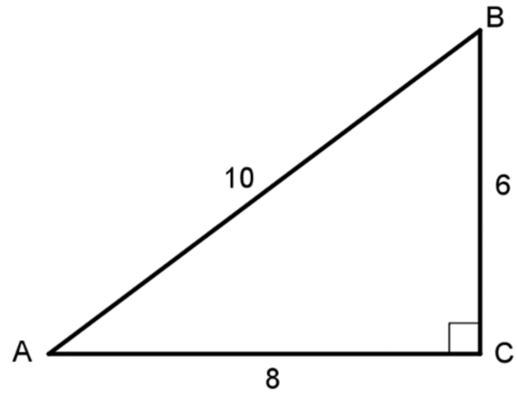
$\sec A = \frac{hypotenuse}{adjacent} =$

$\cot A = \frac{adjacent}{opposite} =$

$\csc B =$

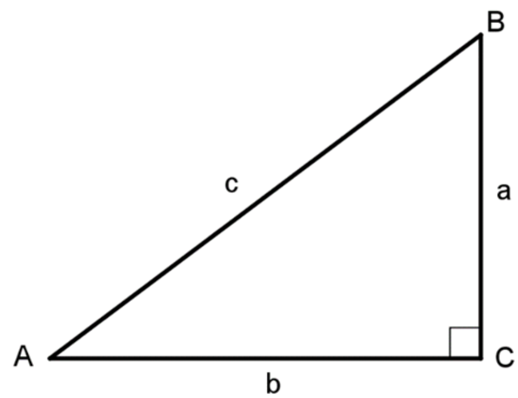
$\sec B =$

$\cot B =$



2. Do the same for this triangle:

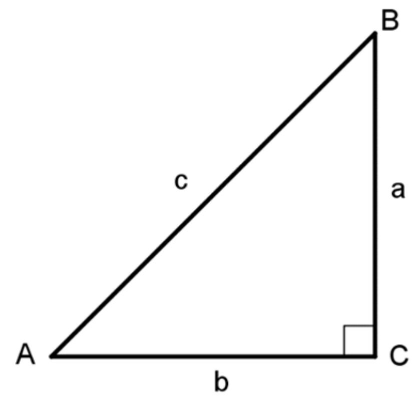
$\sin A =$	$\csc A =$
$\cos A =$	$\sec A =$
$\tan A =$	$\cot A =$
$\sin B =$	$\csc B =$
$\cos B =$	$\sec B =$
$\tan B =$	$\cot B =$



3. Use the information above to write observations you notice that incorporate the **<u>new</u>** trigonometric functions.

4. Do you think these observations will always hold true? Why or why not?

The following is a list of conjectures made by students about right triangles and all six trigonometric relationships. For each, state whether you think the conjecture is true or false. Justify your answer.



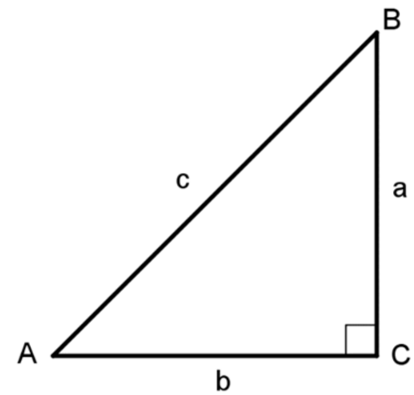
5. $\csc B = \frac{1}{\sin B}$

6. $\cot B = \frac{\sin A}{\cos A}$

7. $\sec A = \cos(90° - A)$

8. $\tan B = \cot(90° - B)$

9. $1 - \sin^2 A = \cos^2 A$



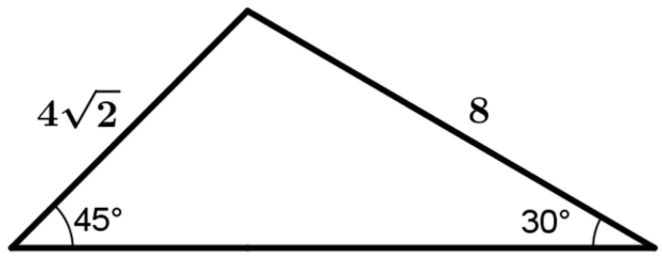
10. $\sec A \cdot \cot A = \csc A$

11. $\sin A \cdot \csc A + \cos A \cdot \sec A = 1$

12. $\tan^2 A + 1 = \sec^2 A$

5.7 Warm Up
Super Special Triangle

Find the area of the triangle below.



5.7 Relationships with Meaning (A Secret Identity)
A Solidify Understanding Task

Simplify each trigonometric expression to a single term.

1. $\cot A \cdot \sec A \cdot \sin A$

2. $\sin A + \cot A \cos A$

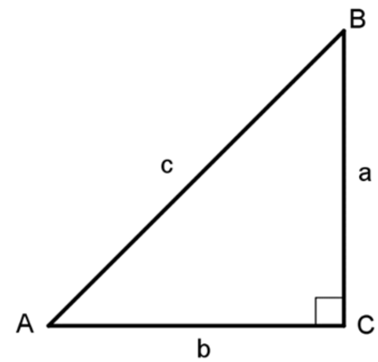
3. $(\cot A - \csc A)\,(\cos A + 1)$

4. $\cos^3 A + \sin^2 A \cos A$

5. $\frac{1+\csc A}{\cos A+\cot A}$

Rewriting the Pythagorean identity, $\sin^2 A + \cos^2 A = 1$, in terms of other trigonometric functions.

6. a. Use the given triangle to write the Pythagorean Theorem.



   b. Divide each term in the Pythagorean Theorem by b^2.

   c. Replace the equation you created in part b with the appropriate trigonometric function(s) to write a new version of the Pythagorean identity.

7. a. Use the same triangle to write the Pythagorean Theorem.

   b. Divide each term in the Pythagorean Theorem by a^2.

   c. Replace the equation you created in part b with the appropriate trigonometric function(s) to write a third version of the Pythagorean identity.

5.8 Warm Up
Simplifying Trigonometric Expressions

Simplify each expression into a single term.

1. $\cos\theta \csc\theta\,(\sec^2\theta - 1)$

2. $\sec\theta \cot\theta - \cot\theta \cos\theta$

3. $\frac{\sin^2\theta \cot^2\theta}{1-\sin^2\theta}$

4. $\frac{\cos\theta}{1+\sin\theta} + \frac{\cos\theta}{1-\sin\theta}$

5.8 Identity Verification
A Practice Understanding Task

The following equations are true for any value of the variable. An equation that is always true is called an identity. Use what you have learned to verify the following identities are true.

1. $\cos\theta\,(\sec\theta - \cos\theta) = \sin^2\theta$

2. $\tan^2\theta - \sec^2\theta = -1$

3. $\frac{1}{\cos^2\theta} - \frac{1}{\cot^2\theta} = 1$

4. $\frac{\tan^2\theta}{\sec\theta+1} + 1 = \sec\theta$

5. $\frac{1}{1-\sin^2 A} = 1 + \tan^2 A$

6. $\frac{\cos A}{1-\sin A} = \sec A + \tan A$

7. $\frac{2+\tan^2 A}{\sec^2 A} - 1 = \cos^2 A$

8. $\frac{\sin^2 A + 2\sin A + 1}{1-\sin^2 A} = \frac{\sin A + 1}{1-\sin A}$

Integrated Math 3
Module 5
Modeling with Geometry
Ready, Set, Go! Homework

Adapted from

The Mathematics Vision Project:
Scott Hendrickson, Joleigh Honey, Barbara Kuehl,
Travis Lemon, Janet Sutorius

Name **Modeling with Geometry | 5.1**

Ready, Set, Go!

Ready

Topic: Comparing perimeter, area, and volume.

Calculate each value. Make certain you label the units for each of your answers.

1. Calculate the perimeter of a rectangle whose dimensions measure 5 cm by 12 cm.

2. Calculate the area of the same rectangle as in question 1.

3. Calculate the volume of a rectangular box whose base is 5 cm by 12 cm and whose height is 8 cm.

4. Look back at questions 1–3. Explain how the units change for each answer.

5. Calculate the surface area for the box in Question 3. Assume it does NOT have a lid. Identify the units for the surface area. How do you know your units are correct?

6. Calculate the circumference of a circle whose radius measures 8 inches.

7. Calculate the area of the circle in question 6.

8. Calculate the volume of a ball with a diameter of 16 inches. $\left(V = \frac{4}{3}\pi r^3\right)$

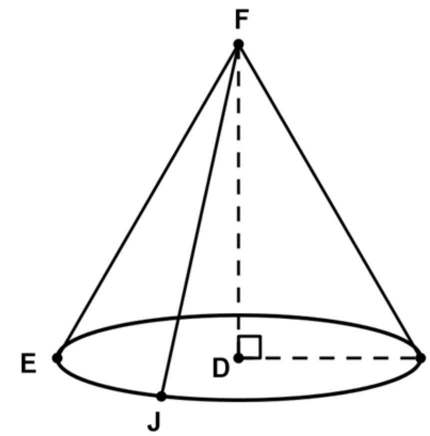
9. Calculate the surface area of the ball in question 8. $(SA = 4\pi r^2)$

10. If a measurement were given, explain how you know if it represented a perimeter, an area, or a volume.

11. Which type of measurement in the questions above, would be considered a “linear measurement?” Explain how you know these are “linear measurements.”

Set

Topic: Cross sections of a cone



Consider the intersection of a plane and a cone.

12. If the plane were parallel to the base of the cone, what would be the shape of cross section? Can you think of 2 possibilities? Explain.

13. If the plane intersected the cone on a slant, so that it intersected segment $\overline{EF}$ and circle D, what would be the shape of the cross section?

14. Describe how the plane would need to intersect the cone in order to get a cross section that is a triangle. Would the triangle be scalene, isosceles, or equilateral? Explain.

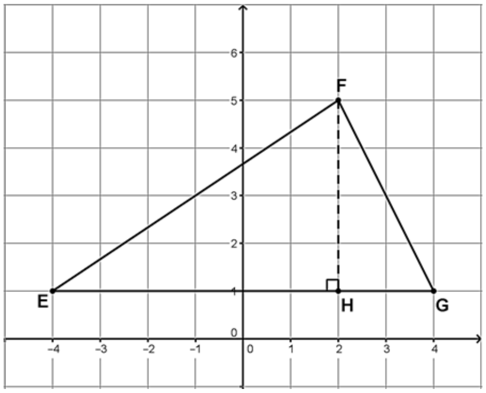
15. Would it be possible for the intersection of a plane and a cone to be a line? Explain.

Go

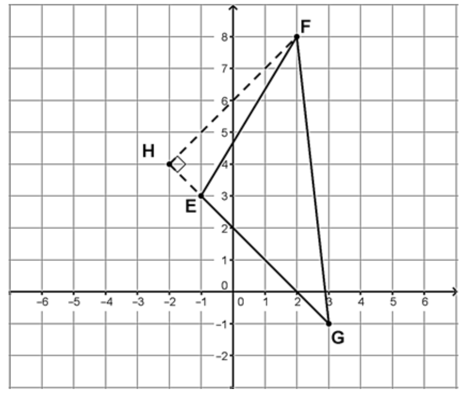
Topic: Area of a triangle

Calculate the area of each triangle.

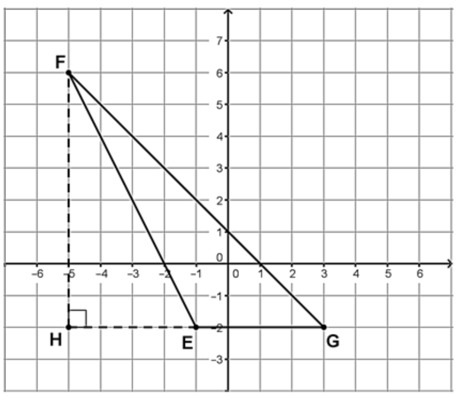
16.



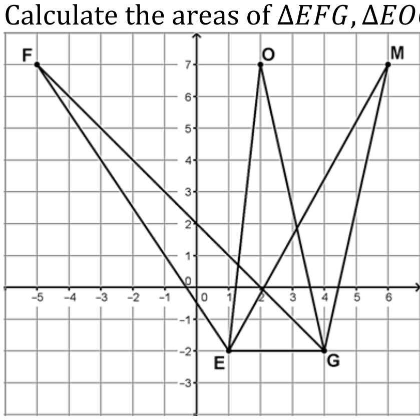
17.



18.



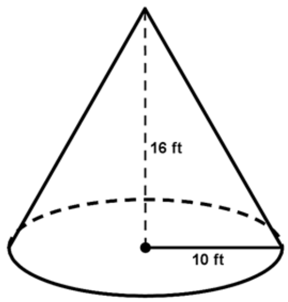
19. Calculate the areas of ΔEFG, ΔEOG, and ΔEMG. Justify your answers.



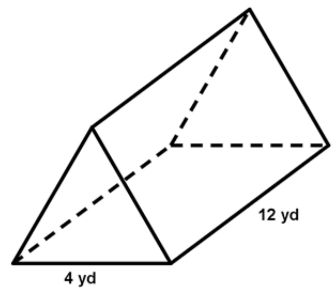
Topic: Finding volume of solids.

Find the volume of each solid.

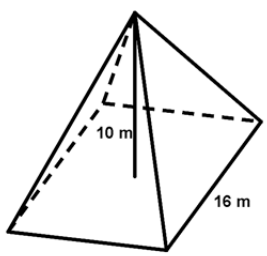
20. Cone



21. Equilateral triangular prism



22. Pyramid



Topic: Finding area of regular polygons.

Find the area of each regular polygon.

23. A regular hexagon with radius of 6 cm.

24. A regular decagon with side length of 8 ft.

Name **Modeling with Geometry | 5.2**

Ready, Set, Go!

Ready

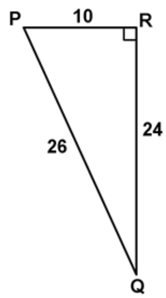
Topic: Finding the trigonometric ratios in a triangle

Use the given measures on the triangles to write the indicated trigonometry value. Leave answers as simplified fractions.

1. $\sin P =$

   $\cos P =$

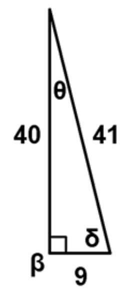
   $\tan P =$



2. $\sin \theta =$

   $\cos \theta =$

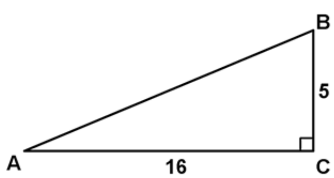
   $\tan \theta =$



3. $\sin B =$

   $\cos B =$

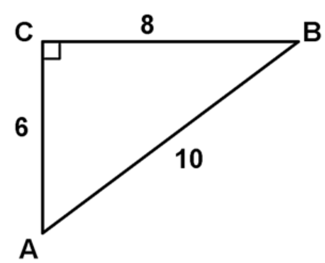
   $\tan B =$



4. $\sin A =$

   $\cos A =$

   $\tan A =$

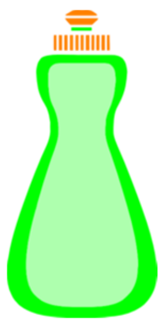


Set

Topic: Solids of revolution

For each of the following solids, draw the two-dimensional shape that would be revolved about the *x*-axis to generate it.

5.

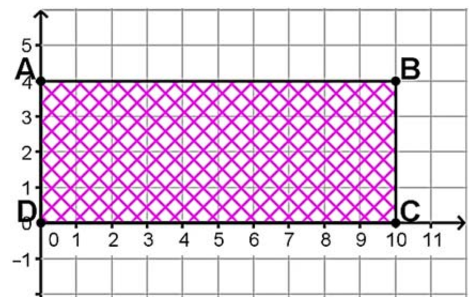


6.

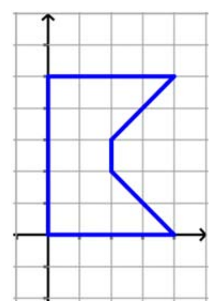
7.

8.

9. Name something in your house that would be shaped like the solid of revolution formed, if the figure on the right were rotated about the *x*-axis.



10. Name something in the world would be shaped like the solid of revolution formed if the figure on the right were rotated about the *y*-axis.

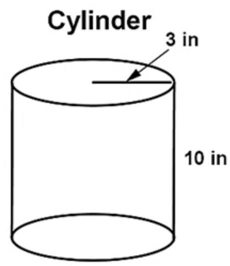


Go

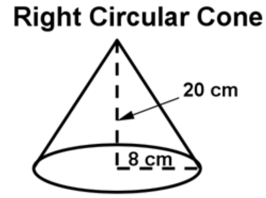
Topic: Using formulas to find the volume of a solid.

Find the volume of the indicated solid.

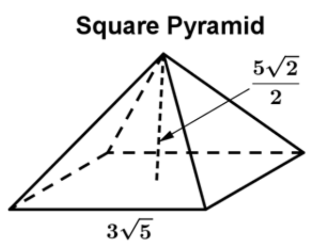
11. $V = \pi r^2 h$
    $r = 3\ in$
    $h = 10\ in$



12. $V = \frac{1}{3}Bh$
    $r = 8\ cm$
    $h = 20\ cm$

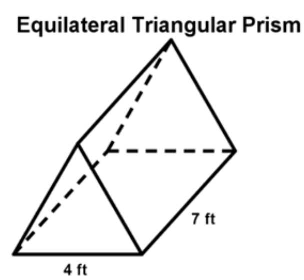


13. $V = \frac{1}{3}Bh$
    $h = \frac{5\sqrt{2}}{2}\ m$
    base edge $= 3\sqrt{5}\ m$



14. $V = Bh$

    $h = 7\ ft$
    base edge $= 4\ ft$



Name **Modeling with Geometry | 5.3**

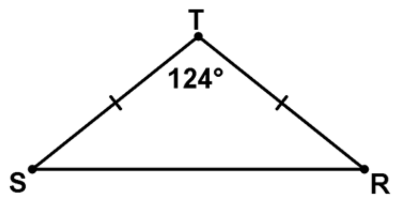
Ready, Set, Go!

Ready

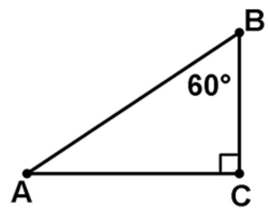
Topic: Finding missing angles in triangles

Use the given information and what you know about triangles to find the missing angles. All angle measures are in degrees.

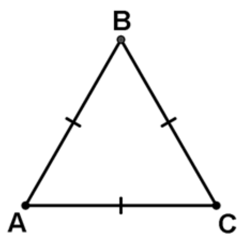
1.



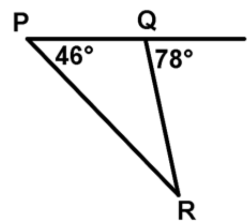
2.



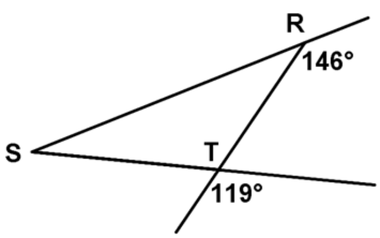
3.



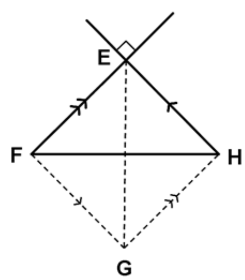
4.



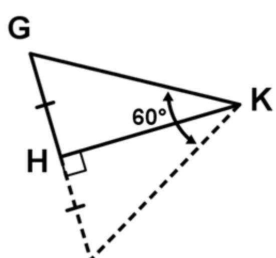
5.



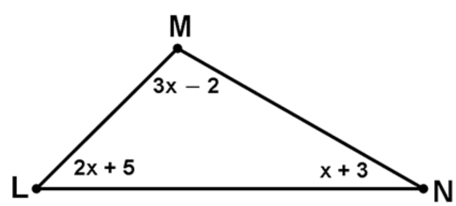
6. $\overline{EG} \cong \overline{FH}$



7.

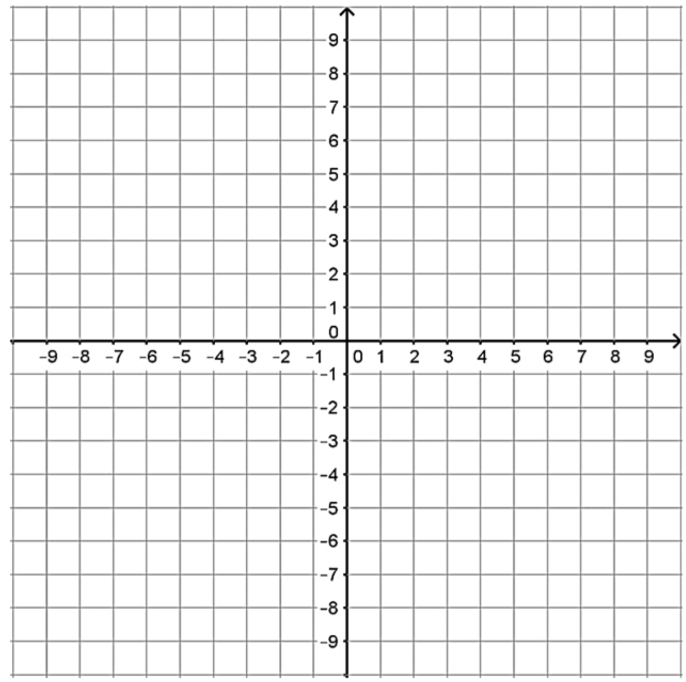


8.



Set

Topic: Finding the surface area and volume of combined shapes

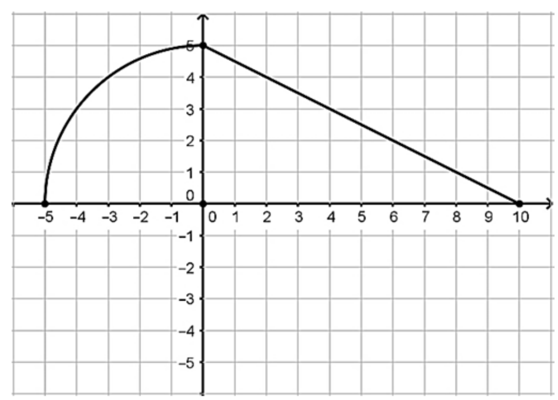


9. a. On the grid at right, sketch the lines $y = 9 - \frac{1}{2}x$ and $y = x$.

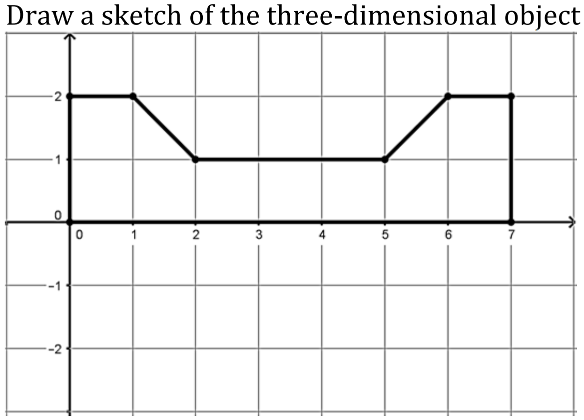
   b. Shade the region formed by the two lines and the y-axis. Find the area of the shaded region.

   c. Find the volume of the solid that is formed when the triangle is rotated rapidly about the y-axis.

10. Find an estimation for volume of the image below, when it is rapidly spun around the x-axis. Use the given points to dissect the shape into smaller pieces.



11. Draw a sketch of the three-dimensional object formed by rotating the figure about the x-axis.

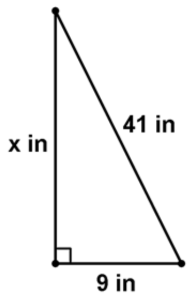


Go

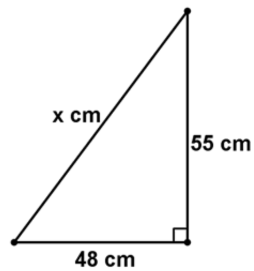
Topic: Solving for the missing side in a right triangle

Calculate the missing side in the right triangles. Give your answers in simplified radical form.

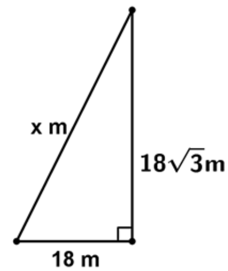
12.



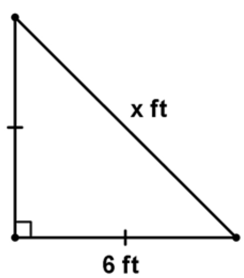
13.



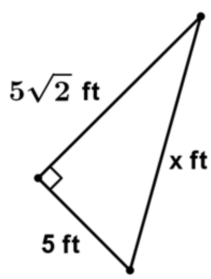
14.



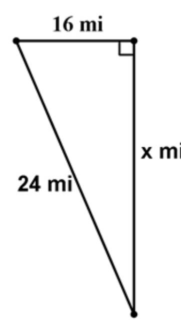
15.



16.



17.



Ready, Set, Go!

Ready

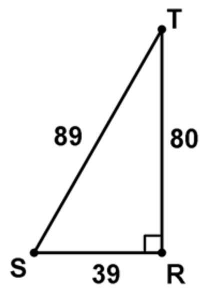
Topic: Finding the trigonometric ratios in a triangle

Use the given measures on the triangles to write the indicated trigonometry value. Write them as reduced fractions.

1. $\sin S =$

   $\cos S =$

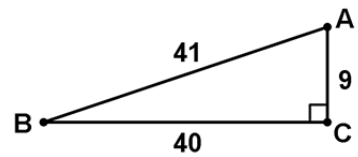
   $\tan S =$



2. $\sin B =$

   $\cos B =$

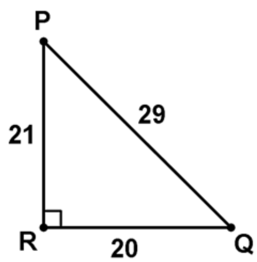
   $\tan B =$



3. $\sin P =$

   $\cos P =$

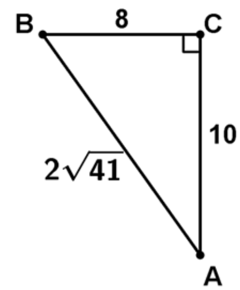
   $\tan P =$



4. $\sin A =$

   $\cos A =$

   $\tan A =$



5. My calculator tells me that $\frac{\sqrt{2}}{2} = 0.7071067812$. Is one value more accurate than the other? Explain.

Set

Topic: Applications of volume, weight, and density

6. The figure at the right is of 2 grain storage silos. The diameter of each silo measures 24 feet and the height of the cylinder measures 51 feet. The height of the cone adds an additional 12 feet. Find the total volume of one silo.

7. How many bushels of grain will each silo be able to store, if a bushel is 1.244 cubic feet? Assume it can be filled to the top.

8. Density relates to the degree of compactness of a substance. A cubic inch of gold weighs a great deal more than a cubic inch of wood because gold is more dense than wood. The density of grains also varies. Use the information below to calculate how many tons of each grain can be stored in one silo. (1 ton = 2000 lbs)

   a. 1 bushel of oats weighs 32 pounds

   b. 1 bushel of barley weighs 48 pounds

   c. 1 bushel of wheat weighs 60 pounds

9. A $\frac{3}{4}$ - ton pickup has the capacity to haul a little more than 1500 lbs. If the hauling bed of the pickup measures 4 ft. by 6.5 ft. by 2 ft., can a $\frac{3}{4}$ - ton pickup safely haul a full (level) load of oats, barley, or wheat? Justify your answer for each type of grain.

Go

Topic: Forms of linear, quadratic, and cubic functions

For each function: Identify the type, discuss the end behavior, discuss any special features including increasing and decreasing intervals, location of max/min values, intercepts, etc. Then graph it.

10. Equation: $f(x) = (x - 2)(x + 3)$

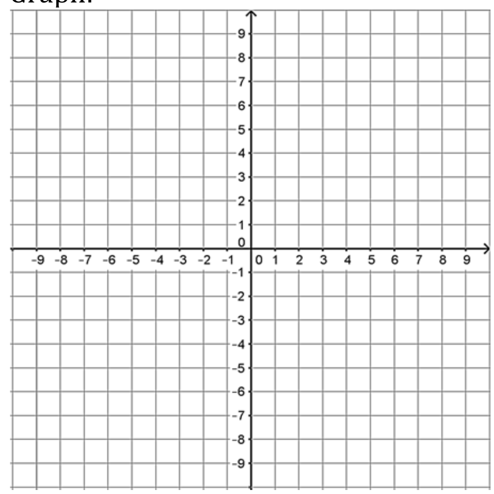
What I know about this function:

End behavior:

as $x \to -\infty, f(x) \to$

as $x \to \infty, f(x) \to$

Graph:



11. Equation: $g(x) = x^2 + 6x + 9$

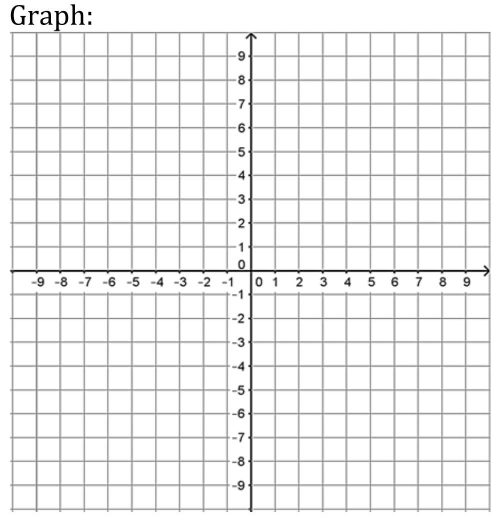
What I know about this function:

End behavior:

as $x \to -\infty, f(x) \to$

as $x \to \infty, f(x) \to$

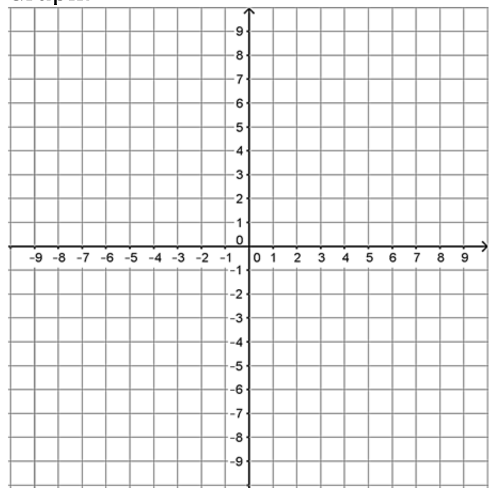
Graph:



12. Equation: $y = (x + 2)(x - 1)(x - 2)$

What I know about this function:

Graph:



End behavior:

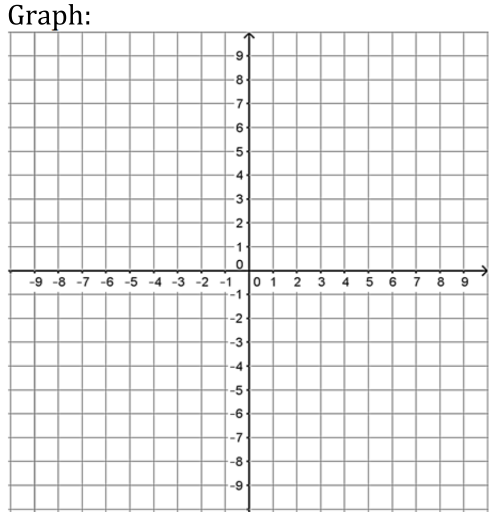
as $x \to -\infty, f(x) \to$

as $x \to \infty, f(x) \to$

13. Equation: $h(x) = 2(x - 5) + 3$

What I know about this function:

Graph:



End behavior:

as $x \to -\infty, f(x) \to$

as $x \to \infty, f(x) \to$

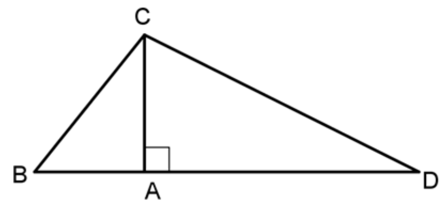
Ready, Set, Go!

Ready

Topic: Finding missing measurements in triangles

Use the given figure to answer the questions. Round your answers to hundredths place. Questions 1-8 all refer to the same triangle below.

Given: $\boldsymbol{m\angle CBD = 51^\circ}$
$\boldsymbol{m\angle CDA = 30^\circ}$
$\boldsymbol{m\angle CAD = 90^\circ}$
$\boldsymbol{CA = 6\ ft}$



1. Find $m\angle BCA$ and $m\angle ACD$

2. Find $m\angle BCD$

3. Find BC

4. Find BA

5. Find CD

6. Find AD

7. Find BD

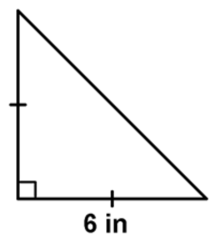
8. Find the area of ΔBCD

Set

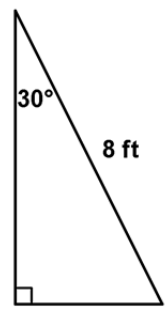
Topic: Triangle relationships in the special right triangles

Fill in all of the missing measures in the triangles. Express answers in simplest radical form.

9.



10.



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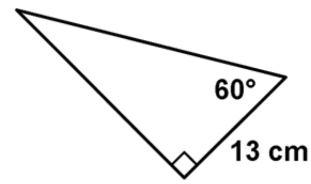
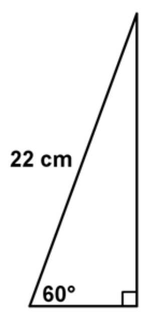


12.



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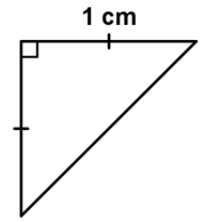
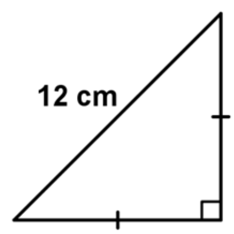
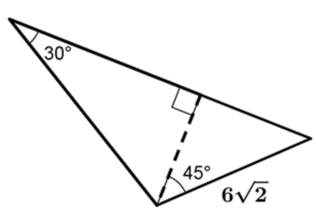


14.



15. Find the exact values of the missing side lengths for the triangle below.



Use an appropriate triangle from questions 9-14 to fill in the function values below.

16.

$\sin 45° =$

$\cos 45° =$

$\tan 45° =$

17.

$\sin 30° =$

$\cos 30° =$

$\tan 30° =$

18.

$\sin 60° =$

$\cos 60° =$

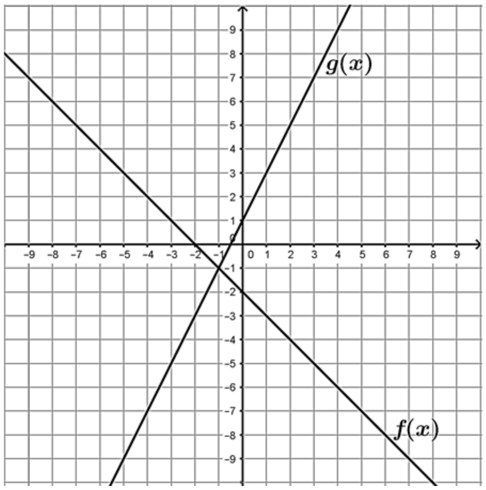
$\tan 60° =$

19. In question 18, does it matter if you used the triangle in question 10, 11, or 12? Explain.

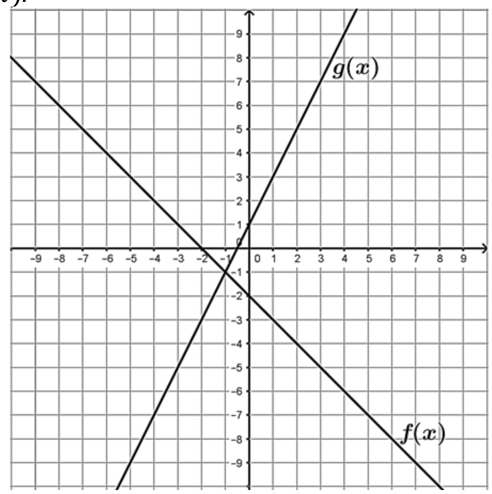
Go

Topic: Function arithmetic

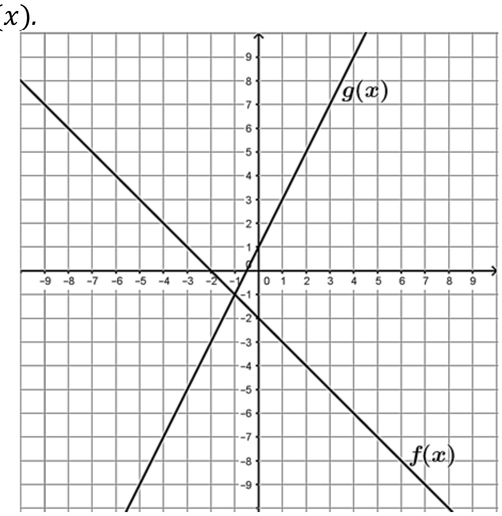
20. Add $f(x)$ and $g(x)$ using the graph below. Draw the new figure on the graph and label it as $s(x)$.



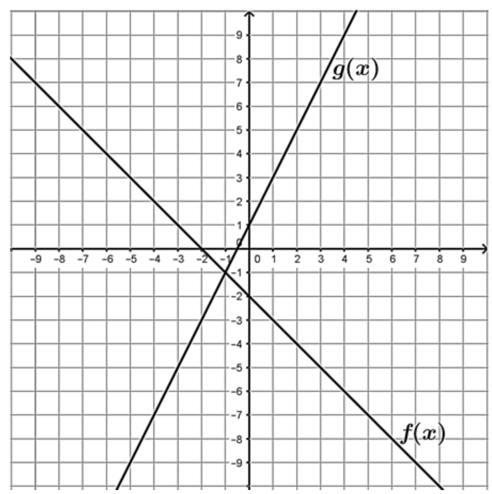
21. Subtract $f(x)$ from $g(x)$ using the graph below. Draw the new figure on the graph and label it as $d(x)$.



22. Multiply $f(x)$ and $g(x)$ using the graph below. Draw the new figure on the graph and label it as $p(x)$.



23. Divide $f(x)$ by $g(x)$ using the graph below. Draw the new figure on the graph and label it as $q(x)$.



24. Write the equations of $f(x)$ and $g(x)$.

25. Write the equation of the sum of $f(x)$ and $g(x)$.
$s(x) =$

26. Write the equation of the difference between $g(x)$ and $f(x)$.
$d(x) =$

27. Write the equation of the product of $f(x)$ and $g(x)$.
$p(x) =$

28. Write the equation of the quotient of $f(x)$ divided by $g(x)$.
$q(x) =$

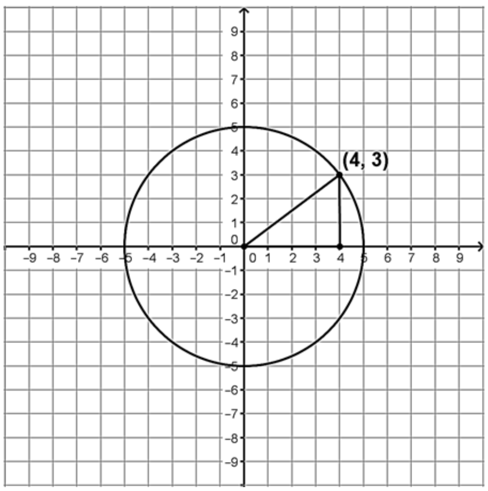
Ready, Set, Go!

Ready

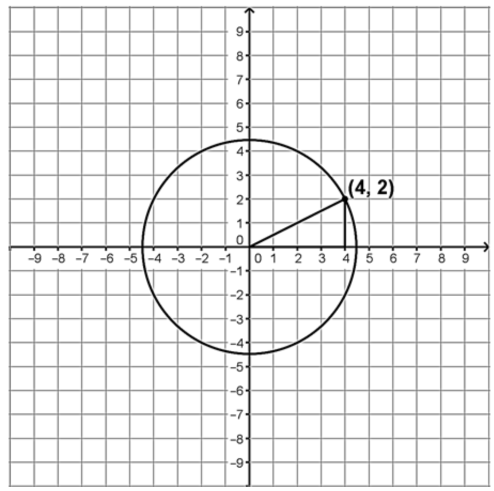
Topic: Finding radius and area of circles.

Find the radius and area of each circle below.

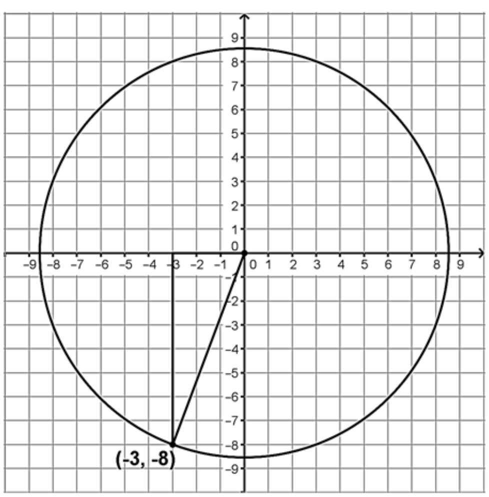
1.



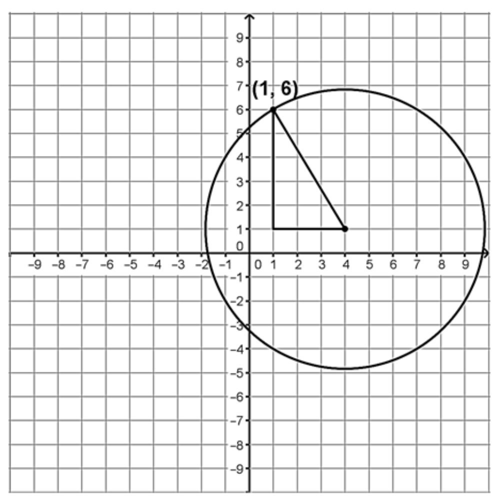
2.



3.



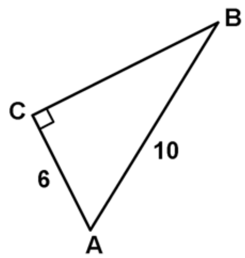
4.



Set

Topic: Trigonometric ratios and connections between them.

5.

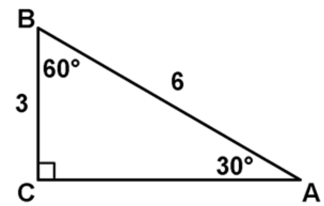


a. $\csc A =$ d. $\csc B =$

b. $\sec A =$ e. $\sec B =$

c. $\cot A =$ f. $\cot B =$

6.



a. $\csc A =$ d. $\csc B =$

b. $\sec A =$ e. $\sec B =$

c. $\cot A =$ f. $\cot B =$

Based on the given trigonometric ratio, sketch a triangle and find a possible value for the missing side as well as the other missing trigonometric ratios. Angle A is one of the two non-right angles in a right triangle.

7. Given: $\sin A = \frac{8}{17}$

a. Sketch of Triangle:

b. Find:

$\cos A =$ $\sec A =$

$\tan A =$ $\cot A =$

$\csc A =$

8. Given: $\csc A = \sqrt{2}$

a. Sketch of Triangle:

b. Find:

$\tan A =$ $\cot A =$

$\cos A =$ $\sec A =$

$\sin A =$

Topic: Trigonometric conjectures

Given a right triangle with angles A and B as the non-right angles, determine if the statements below are true or false. Justify your reasoning and show your argument. Use properties found in class or ratios from a right triangle.

9. $\cos A = \frac{1}{\sin A}$

10. $\tan B = \tan(90° - A)$

11. $\tan A \cdot \cos A = \sin A$

12. $\frac{\sec A}{\csc A} = \tan A$

13. $\frac{\sec A - \cos A}{\sin A} = \tan A$

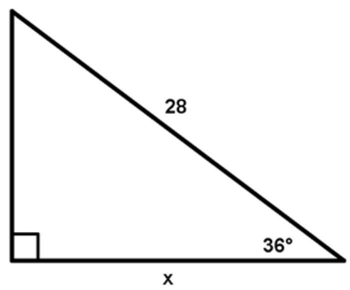
14. $\cot^2 A - 1 = \csc^2 A$

Go

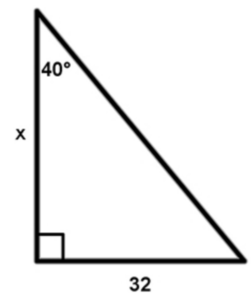
Topic: Finding missing sides of right triangles using trigonometry

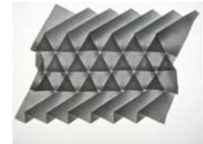
For each right triangle below, write a trigonometric equation and solve it to find the variable. Round to the nearest tenth.

15.



16.





Ready, Set, Go!

Ready

Topic: Using right triangle trigonometry to model situations

For each problem:

- **make a drawing**
- **write an equation**
- **solve and round answers to the nearest hundredth (do not forget to include units of measure)**

1. Jill put a ladder up against the house to try and reach a light that is out and needs to be changed. She knows the ladder is 10 feet long and the distance from the base of the house to the bottom of the ladder is 4 feet. Put a variable on your picture to show how high the ladder reaches.

2. Linda is flying a kite. She lets out 45 yards of string and anchors it to the ground. She determines that the angle between the end of the string and the ground is 58°. Put a variable on the picture that represents the how high off the ground the kite is.

3. Carrie places a 10 foot ladder against a wall. If the ladder makes an angle of 65° with the level ground, how far up the wall is the top of the ladder?

4. In southern California, there is a six mile section of Interstate 5 that increases 2,500 feet in elevation. What is the angle of elevation? Hint: 1 mile = 5,280 feet.

Set

Topic: Simplifying trigonometric expressions

Simplify each trigonometric expression to a single term.

5. $\sin^2\theta \cdot \cot\theta \cdot \csc\theta$

6. $\frac{\tan\theta + \cot\theta}{\cot\theta}$

7. $\frac{\sin\theta(1+\sin\theta)}{1-\cos^2\theta} - 1$

8. $\sec\theta \cdot \frac{\tan\theta}{\tan^2\theta + 1}$

Topic: Finding trigonometric ratios

Find all trigonometric ratios for the triangles described. Hint: draw a right triangle for the given information.

9. Given: A right triangle with the following trigonometric ratio: $\sin(30°) = \frac{1}{2}$, find the exact values for the remaining trigonometric functions.

10. Given: A right triangle with the following trigonometric ratio: $\cot\theta = 1$, find all trigonometric ratios for this triangle.

11. Given: A right triangle with the following trigonometric ratio: $\csc\theta = \frac{3}{2}$, find all trigonometric ratios for this triangle.

Go

Topic: Inverses of functions

Find the inverses of the following functions. If necessary, state the restricted domain that would guarantee that the inverse is a function.

12. $f(x) = 2x^2 - 5$
13. $g(x) = 4 - 2^x$
14. $h(x) = \sqrt{x+1}$
15. $A(x) = 5\log_2(x - 7)$

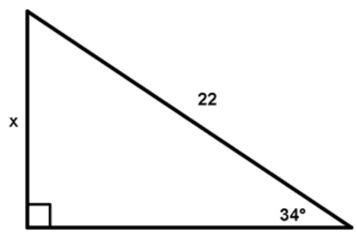
Ready, Set, Go!

Ready

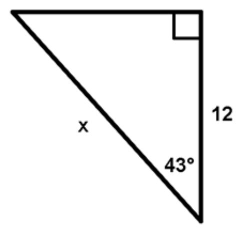
Topic: Solving trigonometric equations.

Write a trigonometric equation to solve for the indicated side. Then find the length of the side to the nearest tenth.

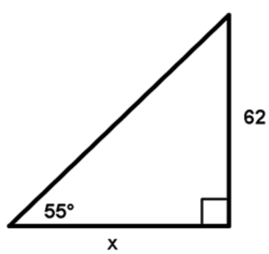
1.



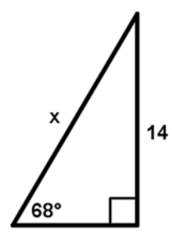
2.



3.



4.

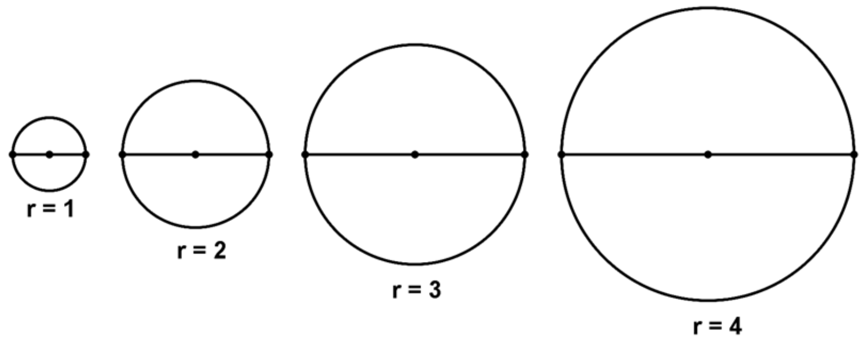


Topic: Radians

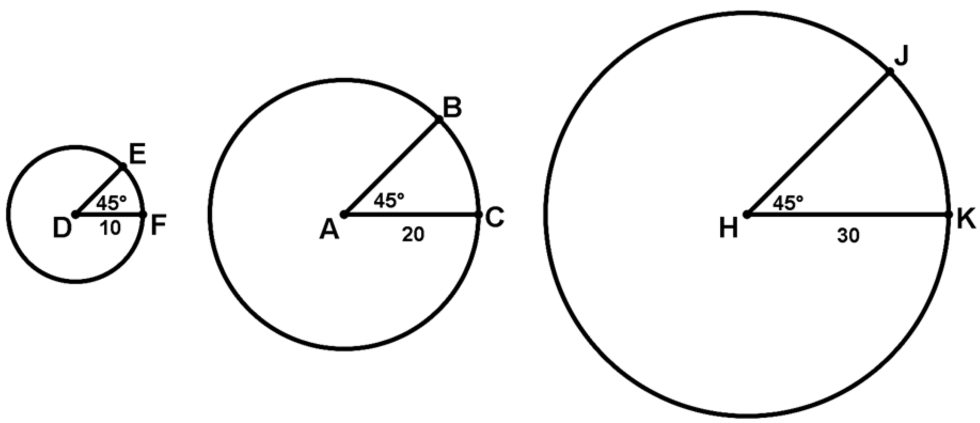
5. Below are circles of radius 1, 2, 3, and 4 units. Each of them has a diameter drawn that cuts them into two equal sectors. Find the arc length of the top half of each of these circles. Then find the radian measure of each arc.

   Radian measure = arc length ÷ radius. Leave answers in terms of π.

Radius	Length of arc for half of the circle	Radian measure of half of the circle
1		
2		
3		
4		



6. There are three circles below each with a different radius. The same size angle 45° has been used to create a sector in each circle. Fill in the table with the length of the arc for the sector, the radian measure and the area of the sector. Leave answers in terms of π.



Radius	Length of arc	Radians	Area of sector
10			
20			
30			

7. Convert the following degree measurements into radians
   a. 30° b. 60° c. 200° d. 540°

Set

Topic: Verifying trigonometric identities

Verify each trigonometric identity to show that the following equations are true for all values of $\boldsymbol{\theta}$.

8. $\cos\theta\,(\sec\theta - \cos\theta) = \sin^2\theta$

9. $\cos\theta + \frac{\sin^2\theta}{1+\cos\theta} = 1$

10. $\frac{\tan^2\theta}{\sec\theta} = \sec\theta - \cos\theta$

11. $2\tan\theta\sec\theta = \frac{1}{1-\sin\theta} - \frac{1}{1+\sin\theta}$

12. $\sin\theta\,(\csc\theta - \sin\theta) = \cos^2\theta$

13. $\frac{\cot^2\theta}{\csc\theta+1} + 1 = \csc\theta$

14. $1 - \sin\theta = \frac{\cos^2\theta}{1+\sin\theta}$

15. $\frac{\sec\theta}{\cos\theta} - \tan^2\theta = 1$

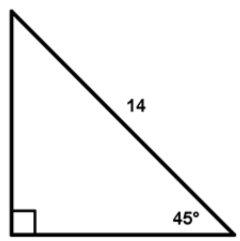
16. $\frac{\sin^2\theta}{1-\cos\theta} = 1 + \cos\theta$

Go

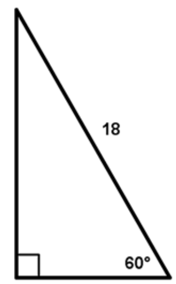
Topic: Special right triangle rules.

Find the missing sides of each right triangle using the $30° - 60° - 90°$ or $45° - 45° - 90°$ right triangle rules. Leave answers with simplified radicals (no decimals).

17.



18.



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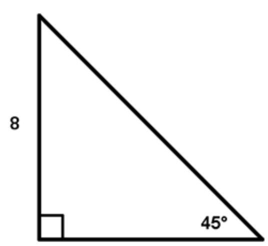

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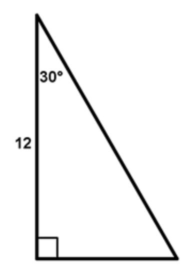


Ready, Set, Go!

Ready
Topic: Rotational symmetry

Hubcaps have *rotational symmetry*. That means that a hubcap does not have to turn a full circle to appear the same. For instance, a hubcap with this pattern, $\otimes$**,** will look the same every $\frac{1}{4}$ turn. ***It is said to have 90° rotational symmetry*** because for each quarter turn it rotates 90°.

State the *rotational symmetry* for the following hubcaps. Answers will be in degrees.

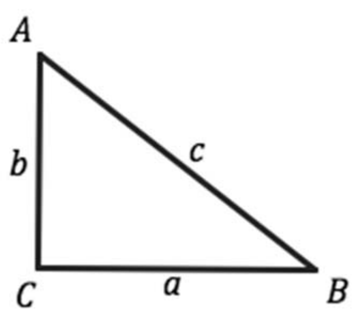
1. 

2. 

3. 

Set
Topic: Trigonometric ratios

4. Use the right triangle below to match an expression in the left column with its equivalent expression in the right column.



A. $\sin A$	a. $\cos A$
B. $\tan A$	b. $\sin^2 B + \cos^2 B$
C. $\sin B$	c. $\tan B$
D. $\frac{1}{\tan A}$	d. $\frac{\sin A}{\cos A}$
E. 1	e. $a^2 + b^2$
F. c^2	f. $\cos B$

Use the given trigonometric ratio to sketch a right triangle and find the missing sides and angles.

5. $\sin A = \frac{1}{2}$

6. $\cos B = \frac{3}{5}$

7. $\cot B = \frac{7}{6}$

8. $\sec A = \frac{10}{7}$

9. $\csc A = \frac{8}{5}$

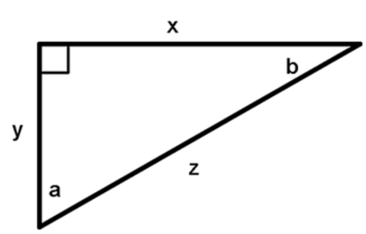
10. $\tan A = \frac{4}{15}$

Go

Topic: Trigonometric ratios

Use the given right triangle to identify the trigonometric ratios and angles were possible.

11.



a. $\sin a =$

b. $\cos a =$

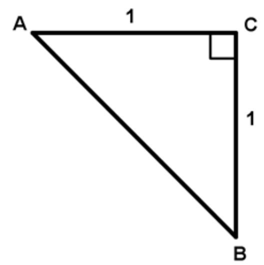
c. $\tan a =$

d. $\sin b =$

e. $\cos b =$

f. $\tan b =$

12.

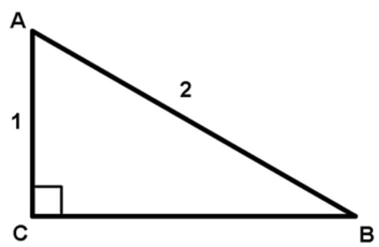


a. $\sin A =$ b. $\sec A =$ c. $\tan A =$

d. $\csc B =$ e. $\cos B =$ f. $\cot B =$

g. $m\angle A =$ h. $m\angle B =$

13.



a. $\csc A =$ b. $\sec A =$ c. $\cot A =$

d. $\sin B =$ e. $\cos B =$ f. $\tan B =$

g. $m\angle A =$ h. $m\angle B =$ i. $m\angle C =$

Topic: Verifying trigonometric identities

Verify each trigonometric identity. Be sure to show all of your steps.

14. $\cot\theta + 1 = \csc\theta\,(\cos\theta + \sin\theta)$

15. $\cos\theta + \sin\theta \cdot \tan\theta = \sec\theta$

16. $\frac{\sin\theta}{\cos\theta+1}+\frac{\cos\theta-1}{\sin\theta}=0$

17. $\tan^2\theta=\csc^2\theta\cdot\tan^2\theta-1$

18. $\sin\theta-\sin\theta\cdot\cos^2\theta=\sin^3\theta$

19. $\sec\theta+\tan\theta=\frac{\cos\theta}{1-\sin\theta}$

20. $\frac{\csc\theta}{\sin\theta}-\frac{\cot\theta}{\tan\theta}=1$

21. $\frac{\cos\theta}{1-\sin\theta}-\frac{\cos\theta}{1+\sin\theta}=2\tan\theta$

Integrated Math 3
Module 6
Trigonometric Functions

Adapted from

The Mathematics Vision Project:
Scott Hendrickson, Joleigh Honey, Barbara Kuehl,
Travis Lemon, Janet Sutorius

Module 6 Overview

Prerequisite Concepts and Skills:

- Finding all six trigonometric ratios from a right triangle
- Translations of functions
- Features of functions (maximum/minimum values, domain, range, intercepts, intervals of increase/decrease)
- Using trigonometry to find angle measures in a right triangle
- Finding side lengths of right triangles using trigonometry
- Using special right triangle rules to find unknown side lengths
- Radian measurement
- Arc length

Summary of the Concepts & Skills in Module 6:

- Graphing all six trigonometric functions
- Modeling using sine and cosine functions
- Solidify understanding of radian measurement
- Introduction to plotting polar coordinates
- Extending right triangle trigonometry definitions to angles of rotation
- Unit circle

Content Standards and Standards for Mathematical Practice Covered:

- Content Standards: F.TF.1, F.TF.2, F.TF.3, F.TF.4, F.TF.5, F.IF.4, F.BF.3, F.BF.4,
- Standards for Mathematical Practice:
  1. Make sense of problems and persevere in solving them.
  2. Reason abstractly and quantitatively.
  3. Construct viable arguments and critique the reasoning of others.
  4. Model with mathematics.
  5. Use appropriate tools strategically.
  6. Attend to precision.
  7. Look for and make use of structure.
  8. Look for and express regularity in repeated reasoning.

Module 6 Vocabulary:

- Circumference
- Angular speed
- Linear speed
- Angle of rotation
- Amplitude
- Period
- Even function
- Arc length
- Polar coordinates
- Coterminal angles
- Inverse trigonometric function
- Sine
- Tangent
- Secant
- Oscillation
- Phase shift
- Standard position
- Initial ray
- Terminal ray
- Periodic function
- Radians
- Odd function
- Rectangular coordinates
- Unit circle
- Reference angle
- Composite function
- Cosine
- Cosecant
- Cotangent

Concepts Used In the Next Module:

- Combinations and compositions of functions
- Dampened sinusoidal functions
- Features of families of functions

Module 6 – Trigonometric Functions

6.1 *Introducing radians as a unit for measuring angles on concentric circles* **(F.TF.1, F.TF.2)**
Classroom Task: Diggin' It – A Develop Understanding Task
Ready, Set, Go Homework: Trigonometric Functions 6.1

6.2 *Using the proportionality relationship of radian measure to locate points on concentric circles* **(F.TF.1, F.TF.2)**
Classroom Task: Staking It – A Solidify Understanding Task
Ready, Set, Go Homework: Trigonometric Functions 6.2

6.3 *Redefining radian measure of an angle as the length of the intercepted arc on a unit circle* **(F.TF.1, F.TF.2)**
Classroom Task: "Sine"ing and "Cosine"ing It – A Solidify Understanding Task
Ready, Set, Go Homework: Trigonometric Functions 6.3

6.4 *Defining sine and cosine on the unit circle in terms of angles of rotation measured in radians* **(F.TF.1, F.TF.2)**
Warm Up: Special Triangles Revisited
Classroom Task: More Unit Circle, Please – A Practice Understanding Task
Ready, Set, Go Homework: Trigonometric Functions 6.4

6.5 *Solving trigonometric equations with and without a restricted domain* **(F.TF.7+)**
Warm Up: What About Tangent?
Classroom Task: How Many Solutions Can There Be? – A Practice Understanding Task
Ready, Set, Go Homework: Trigonometric Functions 6.5

6.6 *Using reference triangles, right triangle trigonometry and the symmetry of a circle to find the y-coordinates of points on a circular path* **(F.TF.5)**
Warm Up: Trigonometric Ratios Return
Classroom Task: George W. Ferris' Day Off – A Develop Understanding Task
Ready, Set, Go Homework: Trigonometric Functions 6.6

6.7 *Using reference triangles, right triangle trigonometry, angular speed and the symmetry of a circle to find the y-coordinates of points on a circular path at given instances in time—an introduction to circular trigonometric functions* **(F.TF.5)**
Warm Up: Angles
Classroom Task: "Sine" Language – A Solidify Understanding Task
Ready, Set, Go Homework: Trigonometric Functions 6.7

6.8 *Extending the definition of sine from a right triangle trigonometry ratio to a function of an angle of rotation* **(F.TF.2)**
Classroom Task: More "Sine" Language – A Solidify Understanding Task
Ready, Set, Go Homework: Trigonometric Functions 6.8

6.9 *Graphing a sine function to model circular motion and relating features of the graph to the parameters of the function* **(F.TF.5, F.IF.4, F.BF.3)**
Classroom Task: More Ferris Wheels – A Solidify Understanding Task
Ready, Set, Go Homework: Trigonometric Functions 6.9

6.10 *Analyzing the graph of $y = \sin x$ and how the angle of rotation affects the value of $y = \sin x$* **(F.TF.2, F.TF.2.1 CA)**
Warm Up: Solving Trigonometric Equations
Classroom Task: Simplifying "Sine" Language – A Solidify Understanding Task
Ready, Set, Go Homework: Trigonometric Functions 6.10

6.11 *Extending the definition of cosine from a right triangle trigonometry ratio to a function of an angle of rotation* **(F.TF.5, F.TF.2)**
Warm Up: The Definition of Cosine
Classroom Task: Moving Shadows – A Practice Understanding Task
Ready, Set, Go Homework: Trigonometric Functions 6.11

6.12 *Introducing the horizontal shift of a trigonometric function in terms of a modeling context* **(F.TF.5, F.BF.3)**
Warm Up: Transformations of Sine and Cosine
Classroom Task: High Noon and Sunset Shadows – A Develop Understanding Task
Ready, Set, Go Homework: Trigonometric Functions 6.12

6.13 *Using trigonometric graphs and inverse trigonometric functions to model periodic behavior* **(F.TF.5, F.BF.4)**
Warm Up: What's Your Angle?
Classroom Task: High Tide – A Solidify Understanding Task
Ready, Set, Go Homework: Trigonometric Functions 6.13

6.14 *Practice using transformations of trigonometric graphs and inverse trigonometric functions to model periodic behavior* **(F.TF.5, F.BF.3, F.BF.4)**
Classroom Task: Getting on the Right Wavelength – A Practice Understanding Task
Ready, Set, Go Homework: Trigonometric Functions 6.14

6.15 *Extending the definition of tangent from a right triangle trigonometry ratio to a function of an angle of rotation, including angles of rotation measured in radians on the unit circle; classifying sine, cosine and tangent functions as even or odd* **(F.TF.2, F.TF.3, F.TF.4, F.IF.5)**
Classroom Task: Off on a Tangent – A Develop and Solidify Understanding Task
Ready, Set, Go Homework: Trigonometric Functions 6.15

6.16 *Graphing reciprocal trigonometric functions* **(F.TF.2.1 CA)**
Warm Up: Reciprocal Trigonometric Values
Classroom Task: Reciprocating the Graphs – A Develop and Solidify Understanding Task
Ready, Set, Go Homework: Trigonometric Functions 6.16

Shifty Functions – A Solidify Understanding Task

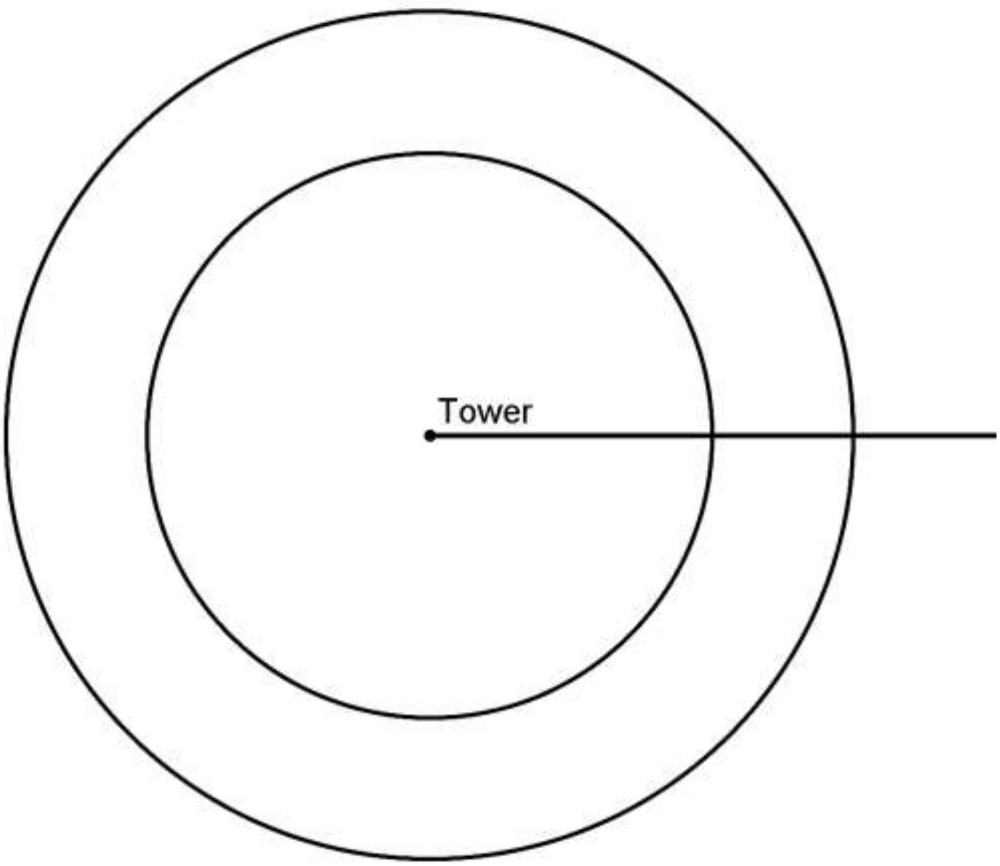
6.1 Diggin' It
A Develop Understanding Task

Alyce, Javier, and Veronica are responsible for preparing a recently discovered archeological site. The people who used to inhabit this site built their city around a central tower, with the main path out of the city heading east (as shown below). The first job of the planning team is to mark the site using stakes so they can keep track of where each discovered item was located.

Part I

1. Alyce suggests that the team place stakes in a circle around the tower, with the distance between the markers on each circle being equal to the radius of the circle. Javier likes this idea but says that using this strategy, the number of markers needed would depend on how far away the circle is from the center tower. Do you agree or disagree with Javier's statement? Explain.

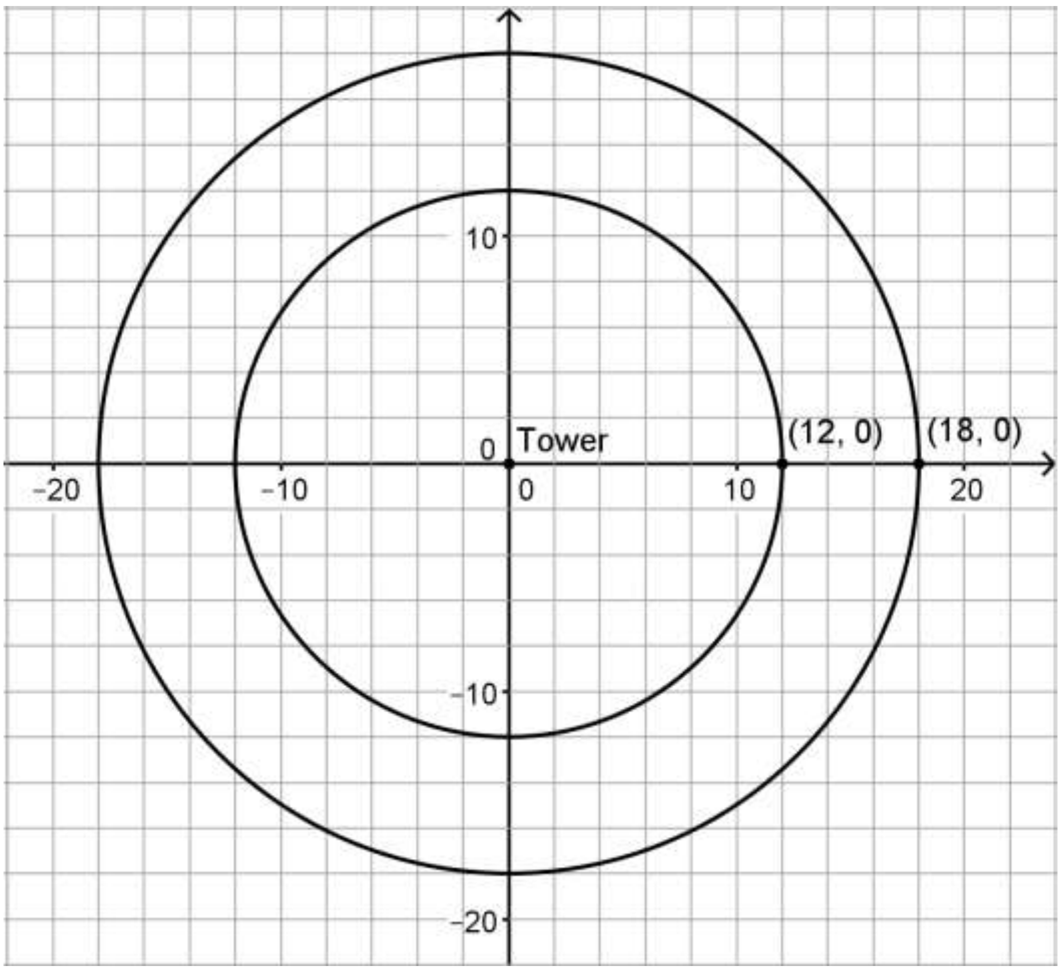
2. Show where the stakes would be located using Alyce's method (from question 1) if one set of markers were to be placed on a circle 12 meters from the center and a second set on a circle 18 meters from the center.



Part II

After looking at the model, Veronica says they need to have more stakes if they intend to be specific with the location of the artifacts. Since most archaeological sites use a grid to mark off sections, Veronica suggests evenly spacing 12 stakes around each circle and using the coordinate grid to label points. Alyce also wants to make sure they record the distance around the circle to each stake. The central tower is located at the origin and the first of each set of 12 stakes for the inner and outer circles is placed at the points $(12, 0)$ and $(18, 0)$, respectively.

3. Your job is to determine the x- and y-coordinates for each of the remaining stakes on each circle, as well as the arc lengths from the points $(12, 0)$ or $(18, 0)$, depending on which circle the stake is located. Keep track of the method(s) you use to find these values and describe each method in your own words. You may want to fill in the table on the next page as you progress work on this question.



Part III

Javier suggests they record the location of each stake and its distance around the circle for the set of stakes on each circle. Veronica suggests it might also be interesting to record the ratio of the arc length to the radius for each circle.

4. Help Javier and Veronica complete this table.

	Inner Circle: r = 12 meters			**Outer Circle: r = 18 meters**		
	Location	**Distance from $(12, 0)$ along circular path**	**Ratio of arc length to radius**	**Location**	**Distance from $(18, 0)$ along circular path**	**Ratio of arc length to radius**
Stake 1	$(12, 0)$	0	0	$(18, 0)$	0	0
Stake 2						
Stake 3						
Stake 4						
Stake 5						
Stake 6						
Stake 7						
Stake 8						
Stake 9						
Stake 10						
Stake 11						
Stake 12						

5. What interesting patterns might Alyce, Javier, and Veronica notice in their work and their table? Summarize any interesting things you have noticed.

6.2 Staking It
A Solidify Understanding Task

After considering different plans for laying out the archeological site described in *Diggin' It,* Alyce, Javier and Veronica have decided to make concentric circles at **10-meter intervals** from the central tower. They have also decided to use **16 stakes per circle**, in order to have a few more points of reference. Using ropes of different lengths to keep the radius constant, they have traced out these circles in the sand. Because they know the circles will soon be worn away by the wind and people's footprints, they feel a sense of urgency to locate the positions of the 16 stakes that will mark each circle. The team wants to be efficient and make as few measurements as possible.

Part I:

Veronica suggests they should locate the stakes around one circle and use those positions to mark where the stakes will go on all of the other circles.

1. What do you think about Veronica's idea? How will marking stake positions on one circle help them locate the positions of the stakes on all of the other circles?

Veronica has decided they should stake out the circle with a radius of 50 meters first. She is standing at the point $(50, 0)$ and knows she needs to move 22 ½° around the circle to place her next stake. But, she wonders, "How far is that?"

- Veronica decides she will find the distance by setting up a proportion comparing degree measurements with lengths.
- Alyce thinks they should find the distance by taking $\frac{1}{16}$ of the circumference.
- Javier wants to start by converting the degree measurement to radians.

2. Show how each team member will calculate this distance. Round to the nearest tenth of a meter.

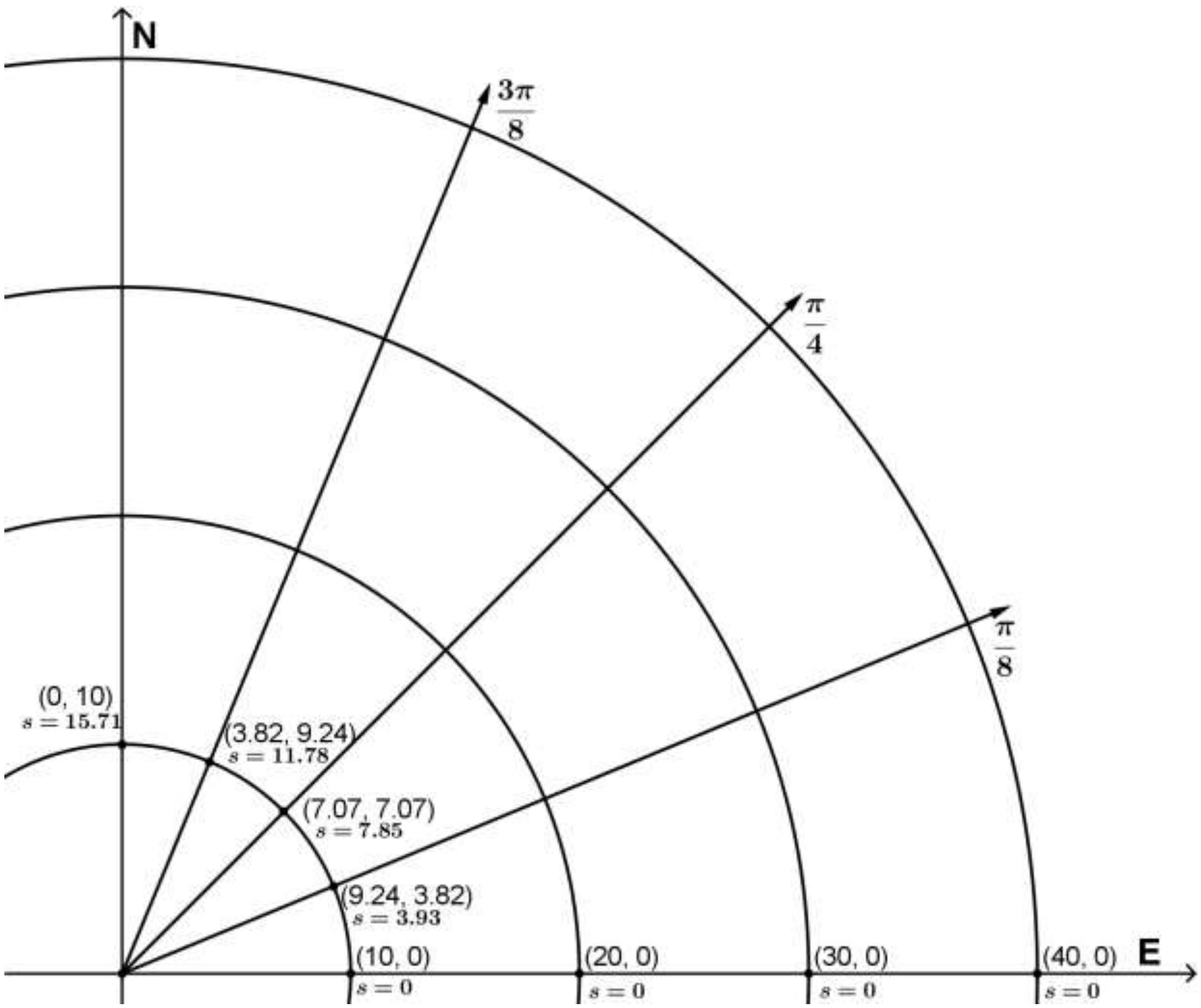
Part II:

Javier has a different idea. He suggests that they should figure out the locations of all of the stakes in quadrant I first, and then find the locations of the stakes in all the other quadrants using these quadrant I locations.

3. What do you think about Javier's suggestion? How will marking the location of all of the stakes in quadrant I (on each quarter circle) help them figure out the location of the stakes in other quadrants?

Javier has already started working on his strategy and has completed the calculations for the 10-meter circle.

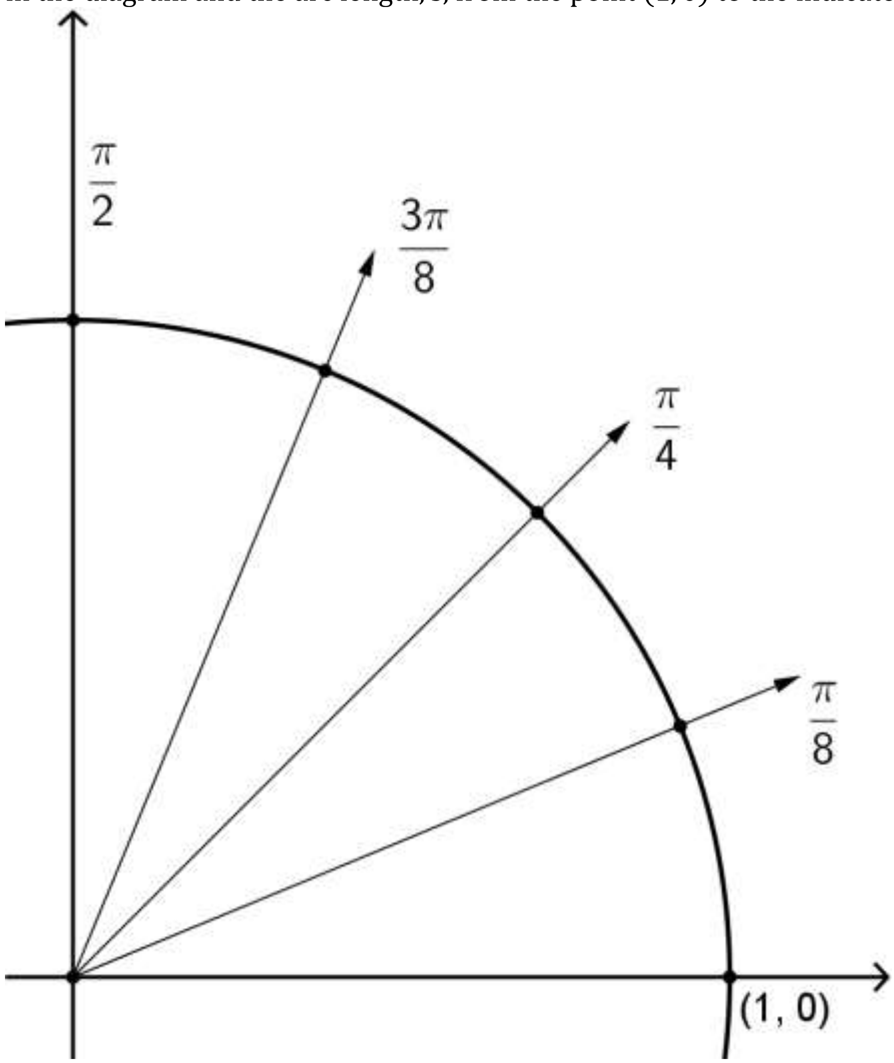
4. Develop a strategy to locate all of the other stakes in the first quadrant for these additional circles. Find the coordinates and arc lengths for each. Describe the strategy you used to make the fewest calculations for finding the coordinates and arc lengths for these additional stakes.



6.3 "Sine"ing and "Cosine"ing It
A Solidify Understanding Task

In the previous tasks you have used the similarity of circles, symmetry of circles, right triangle trigonometry, and proportional reasoning to locate stakes on concentric circles. In this task, we consider points on the simplest circle of all, the circle with a **radius of 1**, which is often referred to as ***the unit circle***.

1. Here is a portion of a unit circle. As in the previous task, *Staking It*, this portion of the unit circle has been divided into intervals measuring $\frac{\pi}{8}$ radians. Find the coordinates of each of the indicated points in the diagram and the arc length, s, from the point $(1, 0)$ to the indicated point.



Javier has been wondering if his calculator will allow him to calculate trigonometric values for angles measured in radians, rather than degrees. He feels like this will simplify much of his computational work when trying to locate the coordinates of stakes on the circles that surround the central tower of the archeological site.

After consulting his calculator's manual, Javier has learned that he can set his calculator in radian mode. After doing so, he is examining the following calculations.

2. With your calculator set in radian mode, find each of the following values. Record your answers as decimal approximations to the nearest thousandth.

$\sin\left(\frac{\pi}{8}\right) =$ $\cos\left(\frac{\pi}{8}\right) =$ $\frac{\pi}{8} =$

$\sin\left(\frac{\pi}{4}\right) =$ $\cos\left(\frac{\pi}{4}\right) =$ $\frac{\pi}{4} =$

$\sin\left(\frac{3\pi}{8}\right) =$ $\cos\left(\frac{3\pi}{8}\right) =$ $\frac{3\pi}{8} =$

$\sin\left(\frac{\pi}{2}\right) =$ $\cos\left(\frac{\pi}{2}\right) =$ $\frac{\pi}{2} =$

The coordinates and arc lengths you found for points on the unit circle seem to be showing up in Javier's computations.

3. Explain why the radian measure of an angle on the unit circle is the same as the arc length.

4. Explain why the **sine** of an angle measured in radians is the same as the ***y*-coordinate** of a point on the unit circle.

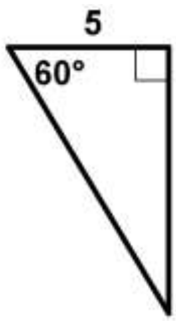
5. Explain why the **cosine** of an angle measured in radians is the same as the ***x*-coordinate** of a point on the unit circle.

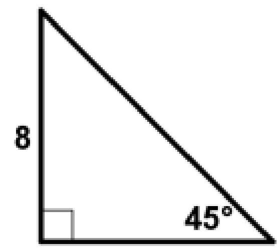
6. Use your results and symmetry to find the following values without using a calculator:

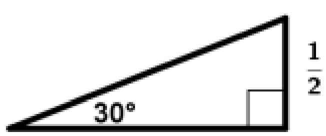
$\sin\left(\frac{5\pi}{8}\right) =$ $\cos\left(\frac{7\pi}{8}\right) =$ $\cos(\pi) =$

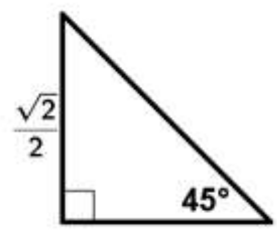
6.4 Warm Up
Special Triangles Revisited

Use ratios for $30° - 60° - 90°$ triangles and $45° - 45° - 90°$ triangles to find the missing sides lengths of the right triangles below. Express all answers in simplest radical form.

1.


2.


3.


4.


6.4 More Unit Circle, Please
A Practice Understanding Task

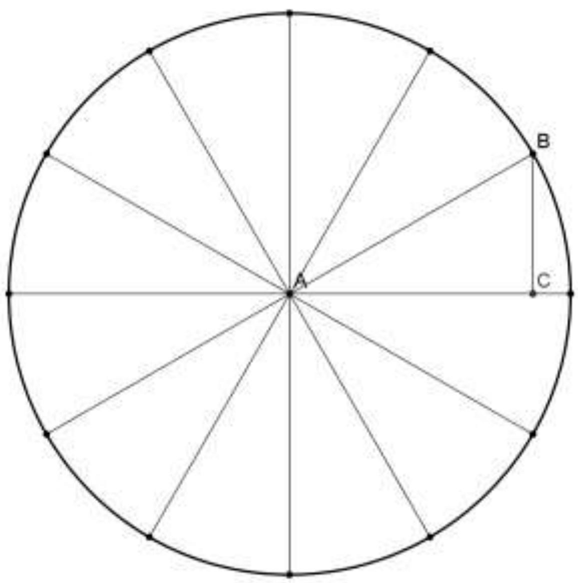
Recall your work in previous tasks on the simplest circle of all, the circle with a **radius of 1**, which is often referred to as ***the unit circle***. You have used the similarity of circles, symmetry of circles, right triangle trigonometry, and proportional reasoning to locate stakes on concentric circles. In this task, we consider other points on the unit circle in order to gain a more thorough understanding.

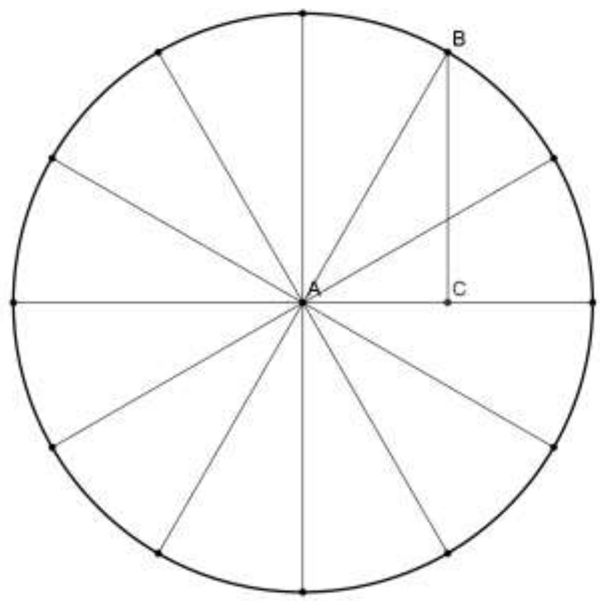
Each unit circle below has been equally divided according to the diagram.

1. Find the measures of $\angle BAC$ and $\angle ABC$ in each diagram.

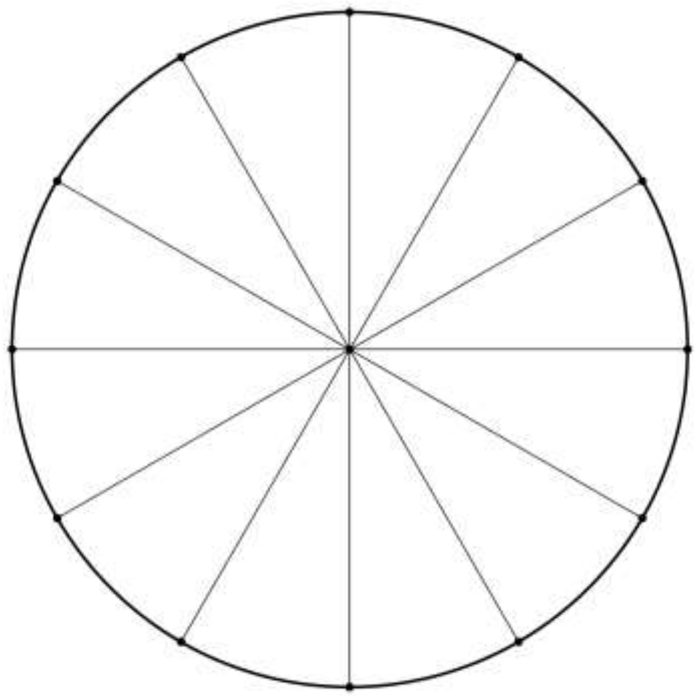
2. Find the exact lengths of $\overline{AB}$, $\overline{AC}$, and $\overline{BC}$. Explain how you found these lengths exactly.

3. Label the exact coordinates of point B in each diagram.





4. Based on your work in questions 1-3, label the exact values of the x- and y-coordinates for each point on the following unit circle. Be sure to label the radian measures on the circle as well.



Use the diagram above to give exact values for the following trigonometric expressions.

5. $\sin\left(\frac{\pi}{6}\right) =$

6. $\sin\left(\frac{5\pi}{6}\right) =$

7. $\cos\left(\frac{7\pi}{6}\right) =$

8. $\sin\left(\frac{\pi}{3}\right) =$

9. $\cos\left(\frac{\pi}{6}\right) =$

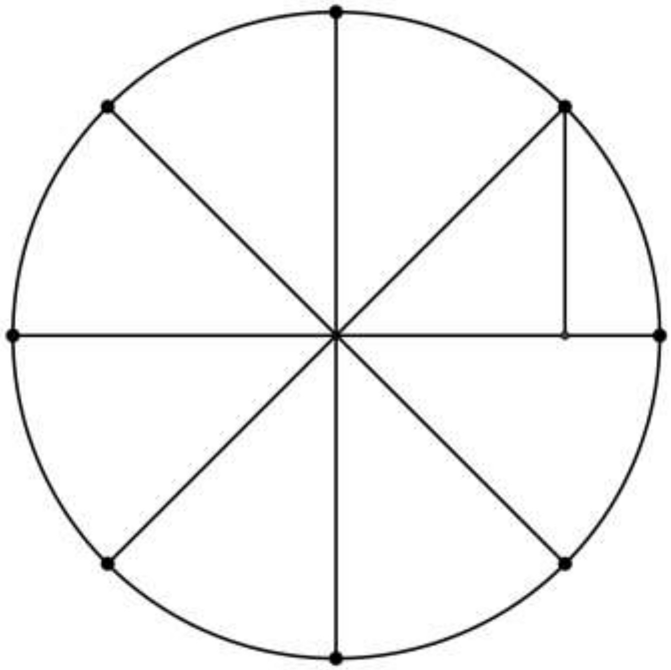
10. $\cos\left(\frac{11\pi}{6}\right) =$

11. $\sin\left(\frac{3\pi}{2}\right) =$

12. $\cos(\pi) =$

13. $\sin\left(\frac{7\pi}{3}\right) =$

14. Here is another unit circle that has been equally divided.



Use the diagram above to give exact values for the following trigonometric expressions.

15. $\sin\left(\frac{\pi}{4}\right) =$

16. $\sin\left(\frac{5\pi}{4}\right) =$

17. $\cos\left(\frac{3\pi}{4}\right) =$

18. $\cos\left(\frac{\pi}{4}\right) =$

19. $\cos\left(-\frac{\pi}{4}\right) =$

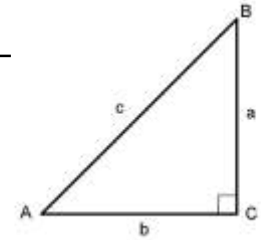
20. $\sin\left(\frac{7\pi}{4}\right) =$

21. $\sin\left(\frac{3\pi}{2}\right) =$

22. $\cos\left(\frac{3\pi}{2}\right) =$

23. $\sin\left(\frac{11\pi}{4}\right) =$

6.5 Warm Up
What About Tangent?



In Math 2, Module 6, you verified the following identity:

$$\tan A = \frac{\sin A}{\cos A}$$

During the last few tasks in this module, you have found connections between $\sin\theta$, $\cos\theta$, and coordinates on the unit circle.

1. Use the above identity to write a conjecture for a relationship between the coordinates on the unit circle and the value of $\tan\theta$ (where θ is an angle of rotation on the unit circle.

2. Use your conjecture to find the exact values of each. Check your results with a calculator.

   a. $\tan\frac{\pi}{4} =$

   b. $\tan\frac{\pi}{3} =$

   c. $\tan\frac{\pi}{2} =$

   d. $\tan\frac{5\pi}{6} =$

6.5 How Many Solutions Can There Be?
A Practice Understanding Task

In previous math courses, you have solved trigonometric equations for an unknown angle, obtaining one or possibly two solutions depending on the quadrant(s) where the trigonometric value existed. Many trigonometric equations have an infinite amount of solutions and some have no solutions. Furthermore, you have explored trigonometric identities, which have solutions for *any* given angle value.

Below are some basic trigonometric equations for you to solve on the interval $[\mathbf{0}, \mathbf{2}\boldsymbol{\pi})$. You may want a unit circle handy in order to obtain all the desired solutions.

1. $\sin x = \frac{\sqrt{3}}{2}$
2. $\cos x = \frac{\sqrt{2}}{2}$
3. $\tan x = -\sqrt{3}$
4. $\tan x = -1$
5. $\cos x = 1$
6. $\sin x\,(1 + \cos x) = 0$

Because of the restricted domain given for the equations above, there were only a **finite number** of solutions for each problem. If there was no restriction on the domain, there would likely be an **infinite amount** of solutions, all evenly spaced. The unit circle can be used as an aid in listing all of the possible solutions.

7. Consider the equation: $\sin x = \frac{\sqrt{2}}{2}$

   a. What are the solutions in the domain $[0, 2\pi)$?

   b. One of the solutions from part a is in the first quadrant on the unit circle. Find four more solutions that are also in the first quadrant if there is no domain restriction. (Hint: think of coterminal angles)

   c. In what quadrant are the other solutions located? List four of these solutions.

   d. Generalize all solutions to the equation $\sin x = \frac{\sqrt{2}}{2}$.

Find all solutions to the following trigonometric equations. Be resourceful in your methods – you may want to consider factoring and using a trigonometric identity along the way.

8. $7\cos x + 9 = -2\cos x$

9. $(\tan x - 1)(\cos x + 1) = 0$

10. $2\sin^2 x - \sin x - 1 = 0$

11. $\sin^2 x - \frac{1}{4} = 0$

12. $\cos x = \frac{\sqrt{3}}{2}$

13. $3\sin^2 x + 7\sin x + 4 = 0$

14. $\sin^2 x = \sin x$

15. $\cos^2 x + 2\cos x + 1 = 0$

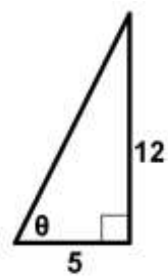
6.6 Warm Up
Trigonometric Ratios Return

Find the trigonometric ratios for the given triangles. Simplify all answers completely.

1. $\sin\theta =$

   $\cos\theta =$

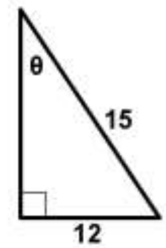
   $\tan\theta =$



2. $\sin\theta =$

   $\cos\theta =$

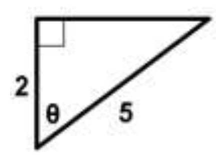
   $\tan\theta =$



3. $\csc\theta =$

   $\sec\theta =$

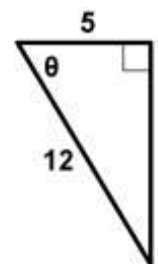
   $\cot\theta =$



4. $\csc\theta =$

   $\sec\theta =$

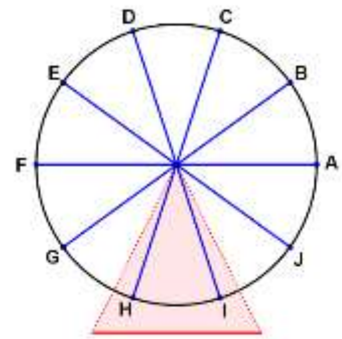
   $\cot\theta =$



6.6 George W. Ferris' Day Off
A Develop Understanding Task

Perhaps you have enjoyed riding on a Ferris wheel at an amusement park. The Ferris wheel was invented by George Washington Ferris for the 1893 Chicago World's Fair.

Carlos, Clarita, and their friends are celebrating the end of the school year at a local amusement park. Carlos has always been afraid of heights, and now his friends have talked him into taking a ride on the Ferris wheel. As Carlos waits nervously in line, he has been able to gather some information about the wheel. By asking the ride operator, he found out that this wheel has a radius of 25 feet and its center is 30 feet above the ground. With this information, Carlos is trying to figure out how high he will be at different positions on the wheel.



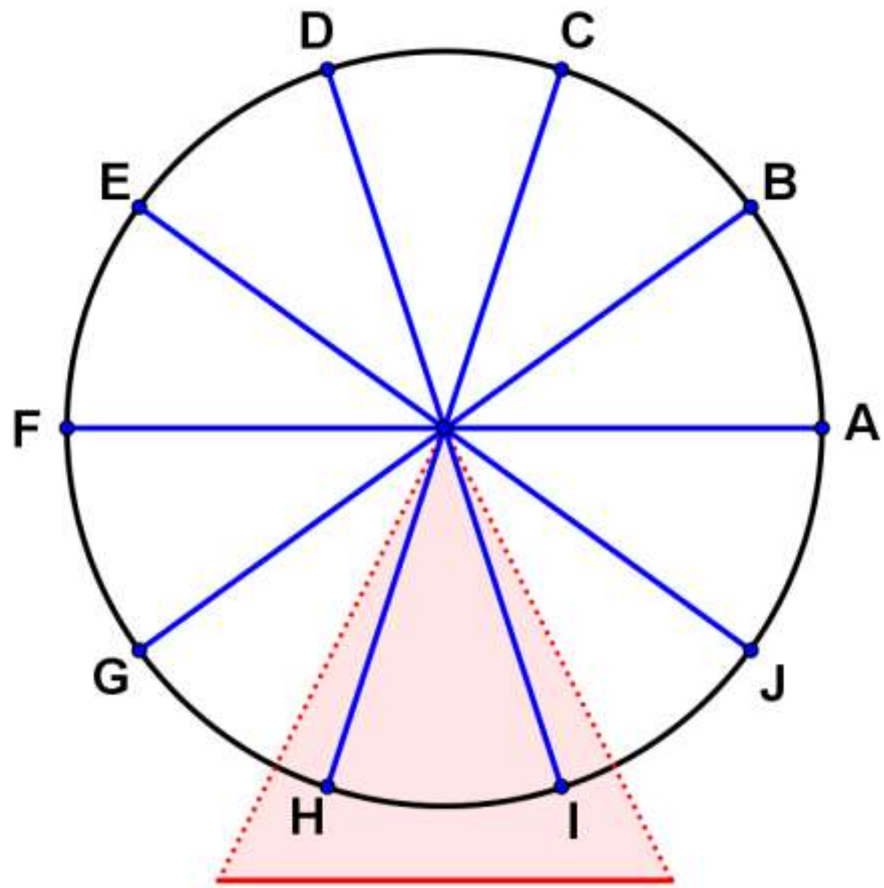
1. How high above the ground will Carlos be when he is at the very top of the wheel?

2. How high will he be when he is at the very bottom of the wheel?

3. How high will he be when he is at the positions farthest to the left or the right on the wheel?

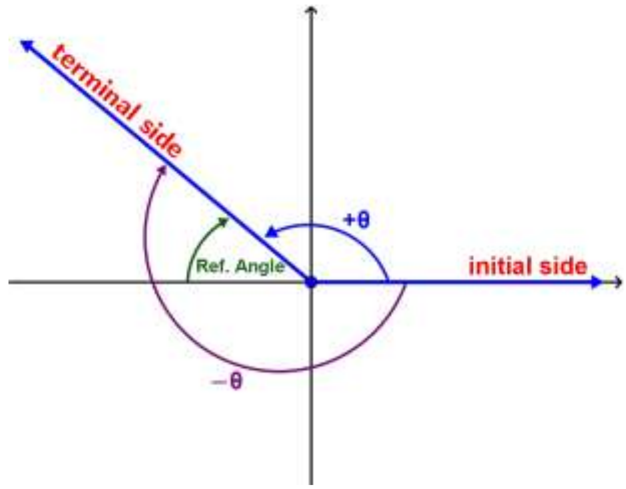
Because the wheel has ten spokes that are evenly spaced, Carlos wonders if he can determine the height of the positions at the ends of each of the spokes as shown in the diagram. Carlos has just finished studying right triangle trigonometry, and wonders if that knowledge can help him.

4. Find the height of each of the points labeled A through J on the Ferris wheel diagram below. Represent your work on the diagram so it is clear to others how you have calculated the height at each point.



6.7 Warm Up
Angles

An angle is in **standard position** if its vertex is located at the origin and one ray is on the positive x-axis. The ray on the x-axis is called the **initial side** and the other ray is called the **terminal side.**



An angle, in standard position, that is rotated counter clockwise has a **positive measurement**. An angle, in standard position, that is rotated clockwise has a **negative measurement**.

If θ is an angle in standard position, its **reference angle** is the acute angle formed by the terminal side of θ and the x-axis.

Two angles, in standard position, are **coterminal** if they share a terminal side.

For each angle measurement below, (A) sketch the angle in standard position, (B) identify the measure of the reference angle, and (C) identify the measure of an angle that is coterminal with the given angle.

1. $110°$
   (A) Sketch:

   (B) Reference Angle:

   (C) Coterminal Angle:

2. $345°$
   (A) Sketch:

   (B) Reference Angle:

   (C) Coterminal Angle:

3. $76°$
   (A) Sketch:

   (B) Reference Angle:

   (C) Coterminal Angle:

4. $-192°$
   (A) Sketch:

   (B) Reference Angle:

   (C) Coterminal Angle:

6.7 "Sine" Language
A Solidify Understanding Task

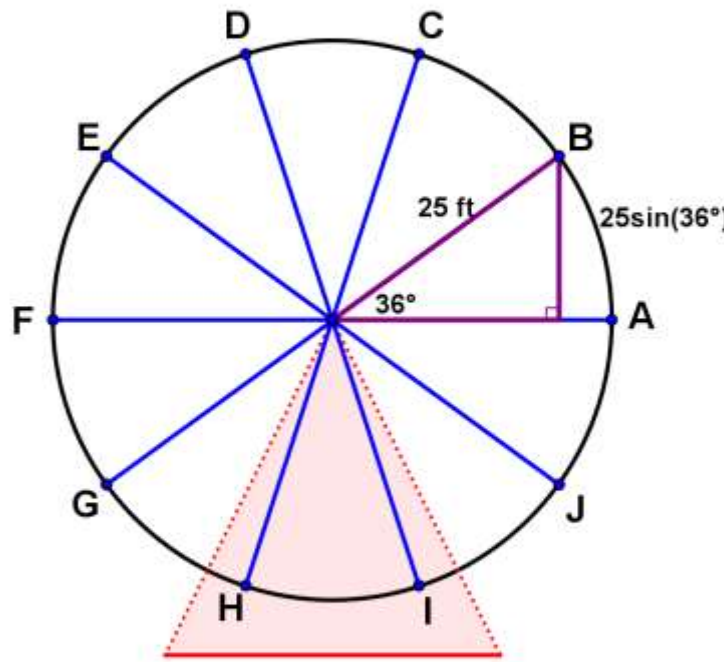
In the previous task, *George W. Ferris' Day Off*, you probably found Carlos' height at different positions on the Ferris wheel using right triangles, as illustrated in the following diagram.

Recall the following facts from the previous task:

- The Ferris wheel has a radius of 25 feet.
- The center of the Ferris wheel is 30 feet above the ground.

Carlos has also been carefully timing the rotation of the wheel and has observed the following additional fact:

- The Ferris wheel makes one complete rotation counterclockwise every 20 seconds.



1. Using this new information, how many degrees does the Ferris wheel rotate per second (angular speed)?

2. How high will Carlos be 2 seconds after passing position A on the diagram?

3. Calculate the height of a rider at each of the following times t, where t represents the number of seconds since the rider passed position A on the diagram. Keep track of any patterns you notice in the ways you calculate the height. As you calculate each height, record the time and height on the diagram as the coordinates
(time, height).

Elapsed time, *t*, since passing position A	**Calculations**	**Height of the rider**
1 sec		
2 sec		
2.5 sec		
3 sec		
4 sec		
5 sec		

4. Examine your calculations for finding the height of the rider during the first 5 seconds after passing position A. During this time, the angle of rotation of the rider is somewhere between 0° and 90°. Write a general formula for finding the height of the rider during this time interval.

5. Calculate the height of a rider at each of the following times *t*, where *t* represents the number of seconds since the rider passed position A on the diagram. Keep track of any patterns you notice in the ways you calculate the height. As you calculate each height, record the time and height on the diagram as the coordinates (time, height).

Elapsed time, *t*, since passing position A	**Calculations**	**Height of the rider**
6 sec		
8 sec		
10 sec		
12 sec		
15 sec		
18 sec		
19 sec		
20 sec		
23 sec		
28 sec		
35 sec		
38 sec		
40 sec		

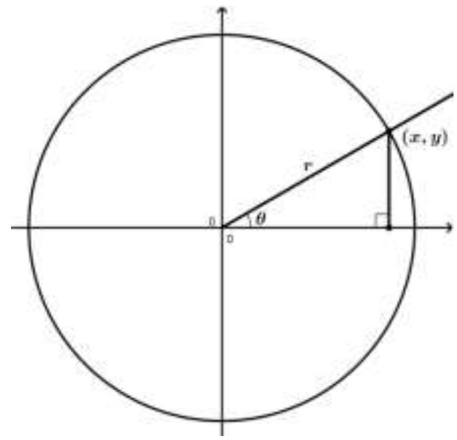
6. How might you find the height of the rider in other "quadrants" of the Ferris wheel, when the angle of rotation is greater than 90°?

6.8 More "Sine" Language
A Solidify Understanding Task

Clarita is helping Carlos calculate his height at different locations around a Ferris wheel. They have noticed that when they use their formula $h(t) = 30 + 25\sin 18t$ their calculator gives them correct answers for the height even when the angle of rotation is greater than 90°. They don't understand why since right triangle trigonometry only defines the sine for acute angles.

Carlos and Clarita decide to ask their math teacher how mathematicians have defined sine for angles of rotation, since the ratio definition no longer holds when the angle isn't part of a right triangle. Here is a summary of that discussion.

> We begin with a circle of radius r whose center is located at the origin on a rectangular coordinate grid. We represent an angle of rotation in ***standard position*** by placing its vertex at the origin, the ***initial ray*** oriented along the positive x-axis, and its ***terminal ray*** rotated θ degrees counterclockwise around the origin when θ is positive and clockwise when θ is negative. Let the ordered pair (x, y) represent the point when the terminal ray intersects the circle. (See the diagram below)
>
> In this diagram, angle θ is between 0 and 90°; therefore, the terminal ray is in quadrant I. A right triangle has been drawn in quadrant I similar to the right triangles we have drawn in the Ferris wheel tasks.



Based on this diagram and the right triangle definition of the sine ratio, find an expression for $\sin\theta$ in terms of the variables *x, y* and *r*.

> $\boldsymbol{\sin\theta =}$

Part I:

Carlos and Clarita are making notes of what they have observed about this new way of defining $\sin\theta$.

Carlos: "For some angles the calculator gives me positive values for $\sin\theta$, and for other angles it gives me negative values."

1. Without using your calculator, list at least five angles of rotation for which the value of $\sin\theta$ produced by the calculator should be **positive**.

2. Without using your calculator, list at least five angles of rotation for which the value of $\sin\theta$ produced by the calculator should be **negative**.

Clarita: "Yeah, and sometimes we can't even draw a triangle at certain positions on the Ferris wheel, but the calculator still gives us values for the sine at those angles of rotation."

3. List possible angles of rotation that Clarita is talking about - positions for which you can't draw a triangle to use as a reference. Then, without using your calculator, give the value of the sine that the calculator should provide at those positions.

Carlos: "And, because of the symmetry of the circle, some angles of rotation should have the same values for the sine."

4. Without using your calculator, list at least five **pairs** of angles that should have the same sine value.

Clarita: "Right! And if we go around the circle more than once, the calculator still gives us values for the sine of the angle of rotation, and multiple angles have the same value of the sine."

Carlos: "Can you have a sine value for an angle less than zero?"

5. List some angles that satisfy Clarita's statement. Explain why her statement is true.

6. Answer Carlos' question. Give specific examples that show your reasoning.

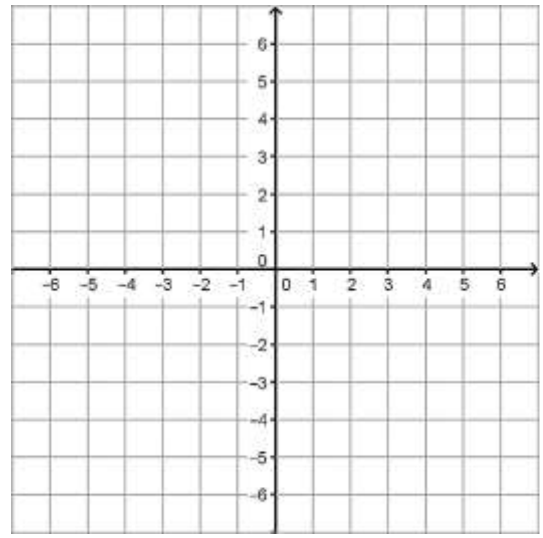
Carlos: "And while we are asking questions, I'm wondering how big or how small the ***value of the sine*** can be as the angles of rotation get larger and larger?"

7. Without using a calculator, what would your answer be to Carlos' question?

Clarita: "Well, whatever the calculator is doing, at least it's consistent with our right triangle definition of sine as the ratio of the length of the side opposite to the length of the hypotenuse for angles of rotation between 0° and 90°."

Part II:
We will use the definition of sine for <u>any</u> angle of rotation. Let's try it out for a specific point on a particular circle.

8. Consider the point $(-3, 4)$, which is on the circle $x^2 + y^2 = 25$



   a. Identify the center and radius of this circle.

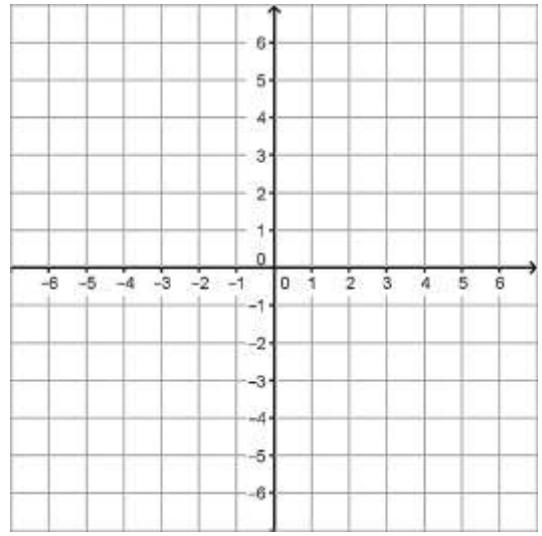
   b. Draw the circle and the angle of rotation to the point $(-3, 4)$. Show and label the initial and terminal ray.

   c. For the angle of rotation you just drew, what would the value of the sine be if we use the definition we wrote for sine at the start of the task?

   d. What is the measure of the angle of rotation? Show how you found this angle of rotation.

   e. Is the calculated value based on this definition the same as the value given by the calculator for this angle of rotation?

9. Consider the point $(-1, -3)$, which is on the circle $x^2 + y^2 = 10$.



a. Identify the center and radius of this circle.

b. Draw the circle and the angle of rotation to the point $(-1, -3)$. Show and label the initial and terminal ray.

c. For the angle of rotation you just drew, what would the value of the sine be if we use the definition we wrote for sine at the start of the task?

d. What is the measure of the angle of rotation? Show how you found this angle of rotation.

e. Is the calculated value based on this definition the same as the value given by the calculator for this angle of rotation?

10. a. For which angles of rotation are the values of sine positive?

b. For which angles of rotation are the values of sine negative?

11. a. Explain how you find the angle of rotation in quadrant II when the reference angle has a measurement of θ.

b. Explain how you find the angle of rotation in quadrant III when the reference angle has a measurement of θ.

c. Explain how you find the angle of rotation in quadrant IV when the reference angle has a measurement of θ.

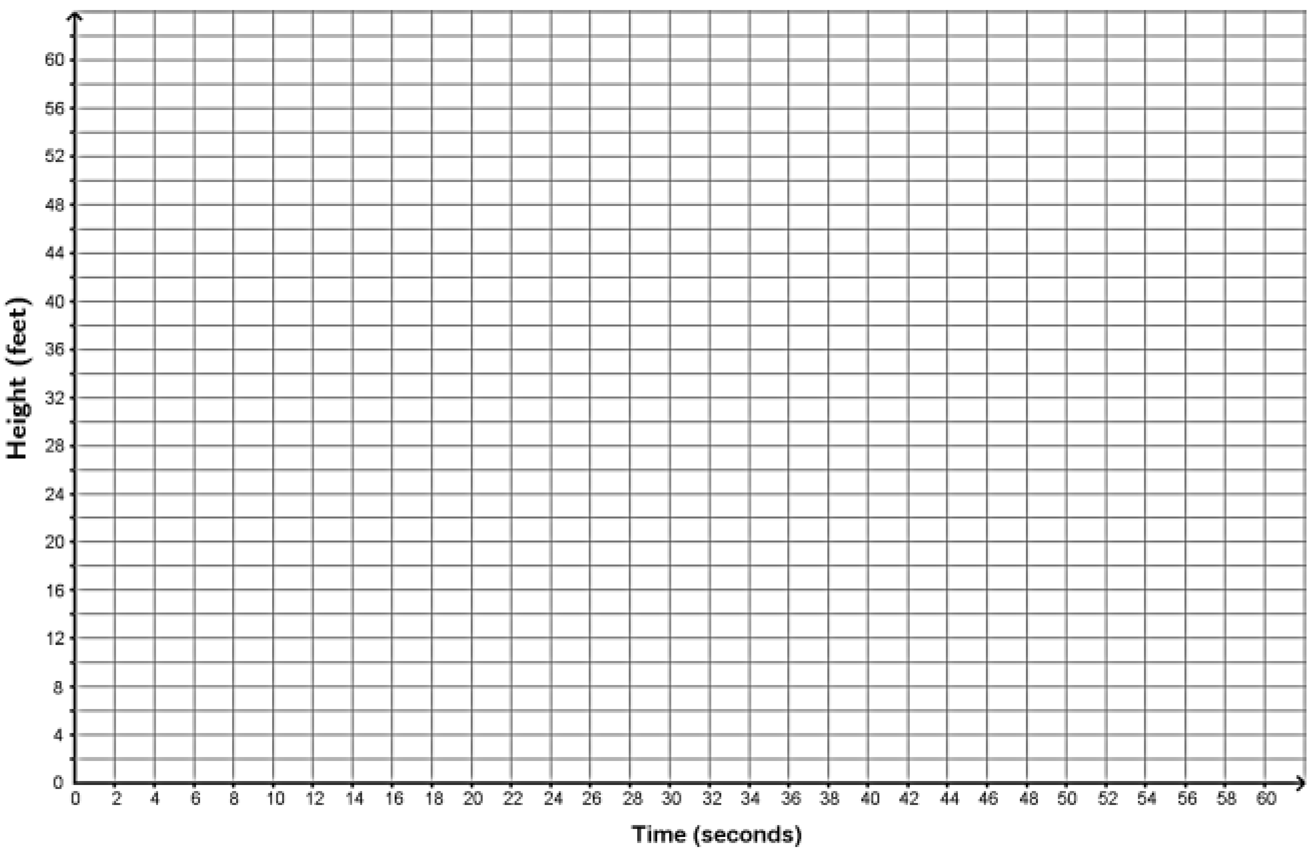
6.9 More Ferris Wheels
A Solidify Understanding Task

In a previous task, *"Sine" Language*, you calculated the height of a rider on a Ferris wheel at different times t, where t represented the elapsed time after the rider passed the position farthest to the right on the Ferris wheel.

Recall the following facts for the Ferris wheel in the previous tasks:

- The Ferris wheel has a radius of 25 feet.
- The center of the Ferris wheel is 30 feet above the ground.
- The Ferris wheel makes one complete rotation counterclockwise every 20 seconds.

1. Based on the data you calculated, as well as any additional insights you might have about riding on Ferris wheels, sketch a graph of the height of a rider on this Ferris wheel as a function of the elapsed time since the rider passed the position farthest to the right on the Ferris wheel. We can consider this position as the rider's starting position at time $t = 0$. Be sure to include the starting position.



2. Write the equation of the graph you sketched in question 1.

3. We began this task by considering the graph of the height of a rider on a Ferris wheel with a radius of 25 feet and center 30 feet off the ground, which makes one revolution counterclockwise every 20 seconds. How would your **graph** change if:

   a. the radius of the wheel was larger? or smaller?

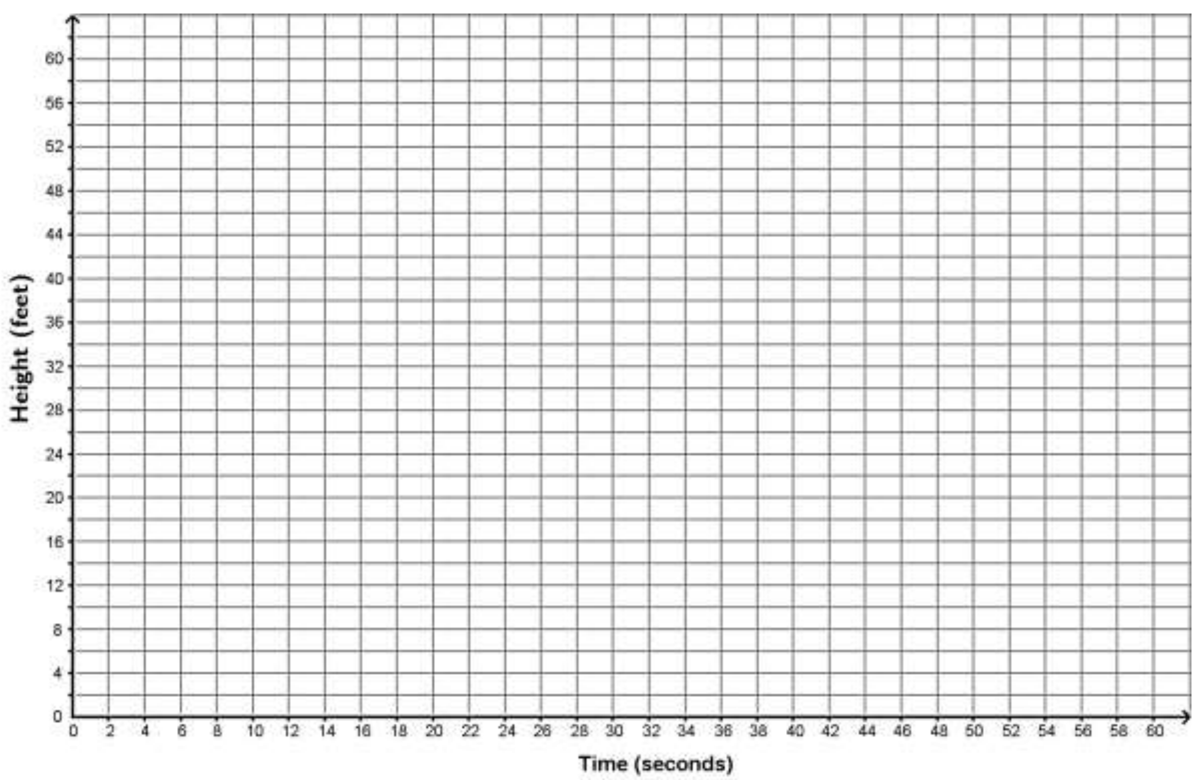
   b. the height of the center of the wheel was greater? or smaller?

   c. the wheel rotates faster? or slower?

4. Of course, Ferris wheels do not all have this same radius, center height, or time of rotation. Describe a different Ferris wheel by changing at least one of the facts listed above.

   Description of my Ferris wheel:

5. Sketch a graph of the height of a rider on your Ferris wheel from question 4 as a function of the elapsed time since the rider passed the position farthest to the right on the Ferris wheel.



6. Write the equation of the graph you sketched in question 5.

7. A general equation for these situations could be written: $y = a \sin bx + d$. What do each of the parameters a, b, and d represent?

8. How does the **equation** of the rider's height change if:

   a. the radius of the wheel is larger? or smaller?

   b. the height of the center of the wheel is greater? or smaller?

   c. the wheel rotates faster? or slower?

9. Write the equation of the height of a rider on each of the following Ferris wheels t seconds after the rider passes the farthest right position.

   a. The radius of the wheel is 30 feet, the center of the wheel is 45 feet above the ground, and the angular speed of the wheel is 15 degrees per second counterclockwise.

   b. The radius of the wheel is 50 feet, the center of the wheel is at ground level (you spend half of your time below ground), and the wheel makes one revolution ***clockwise*** every 15 seconds.

10. Describe the Ferris wheel represented by the equation: $y = 24 \sin 10x + 30$

6.10 Warm Up
Solving Trigonometric Equations

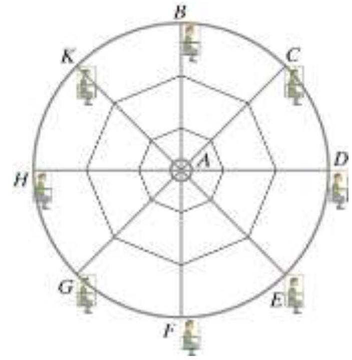
Solve the following equations over the interval $(0, 2\pi]$.

1. $4\cos^2 x - 1 = 0$

2. $2\sin^2 x + 5\sin x - 3 = 0$

6.10 Simplifying "Sine" Language
A Solidify Understanding Task

Part I:
Carlos and Clarita are participating in an afterschool enrichment class where they have to build a model replica of a Ferris wheel.



They are required to use the following measurements:

- The Ferris wheel has a radius of 1 meter.
- The center of the Ferris wheel is at ground level

Carlos has also been carefully timing the rotation of the wheel by placing a doll in the cart in position D and has observed the following additional fact:

- The Ferris wheel makes one complete rotation counterclockwise every 360 seconds

1. Using this new information, how many degrees does the Ferris wheel rotate per second (angular speed)?

2. How high will the doll be 10 seconds after passing position D on the diagram?

3. Calculate the height of the doll at each of the following times t, where t represents the number of seconds since the doll passed position A on the diagram. Keep track of any patterns you notice in the ways you calculate the height. As you calculate each height, record the time and height on the diagram as the coordinates (time, height).

Elapsed time, *t*, since passing position A	**Calculations**	**Height of the doll**
10 sec		
20 sec		
30 sec		
45 sec		
60 sec		
90 sec		

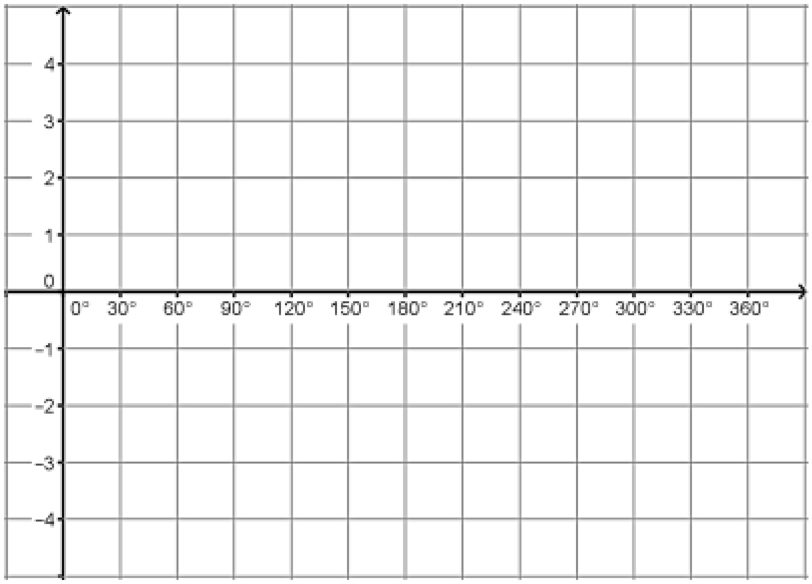
4. Examine your calculations for finding the height of the doll during the first 90 seconds after passing position A. During this time, the angle of rotation of the doll is somewhere between 0° and 90°. Write a general formula for finding the height of the doll during this time interval.

5. Calculate the height of the doll at each of the following times t, where t represents the number of seconds since the doll passed position A on the diagram. Keep track of any patterns you notice in the ways you calculate the height. As you calculate each height, record the time and height on the diagram as the coordinates (time, height).

120 sec		
150 sec		
180 sec		
225 sec		
270 sec		
315 sec		
330 sec		
360 sec		
400 sec		
420 sec		
630 sec		
660 sec		
720 sec		

6. How might you find the height of the doll in other "quadrants" of the Ferris wheel, when the angle of rotation is greater than 90°?

7. Based on the data you calculated, as well as any additional insights you might have about riding on Ferris wheels, sketch a graph of the height of the doll on this Ferris wheel as a function of the elapsed time since the rider passed the position farthest to the right on the Ferris wheel. We can consider this position as the rider's starting position at time $t = 0$. Be sure to include the starting position.



9. Write the equation of the graph you sketched in question 7.

Part II:

Carlos and Clarita are making notes of what they have observed about this new way of defining $\sin\theta$.

Carlos: "For some angles the calculator gives me positive values for $\sin\theta$, and for other angles it gives me negative values."

9. Without using your calculator, list at least five angles of rotation for which the value of $\sin\theta$ produced by the calculator should be **positive**.

10. Without using your calculator, list at least five angles of rotation for which the value of $\sin\theta$ produced by the calculator should be **negative**.

Clarita: "Yeah, and sometimes we can't even draw a triangle at certain positions on the Ferris wheel, but the calculator still gives us values for the sine at those angles of rotation."

11. List possible angles of rotation that Clarita is talking about - positions for which you can't draw a triangle to use as a reference. Then, without using your calculator, give the value of the sine that the calculator should provide at those positions.

Carlos: "And, because of the symmetry of the circle, some angles of rotation should have the same values for the sine."

12. Without using your calculator, list at least five **pairs** of angles that should have the same sine value.

Clarita: "Right! And if we go around the circle more than once, the calculator still gives us values for the sine of the angle of rotation, and multiple angles have the same value of the sine."

Carlos: "Can you have a sine value for an angle less than zero?"

13. List some angles that satisfy Clarita's statement. Explain why her statement is true.

14. a. For which angles of rotation are the values of sine positive?

b. For which angles of rotation are the values of sine negative?

15. a. Explain how you find the angle of rotation in quadrant II when the reference angle has a measurement of θ.

b. Explain how you find the angle of rotation in quadrant III when the reference angle has a measurement of θ.

c. Explain how you find the angle of rotation in quadrant IV when the reference angle has a measurement of θ.

16. Carlos realizes that a Ferris wheel at ground level does not make sense.

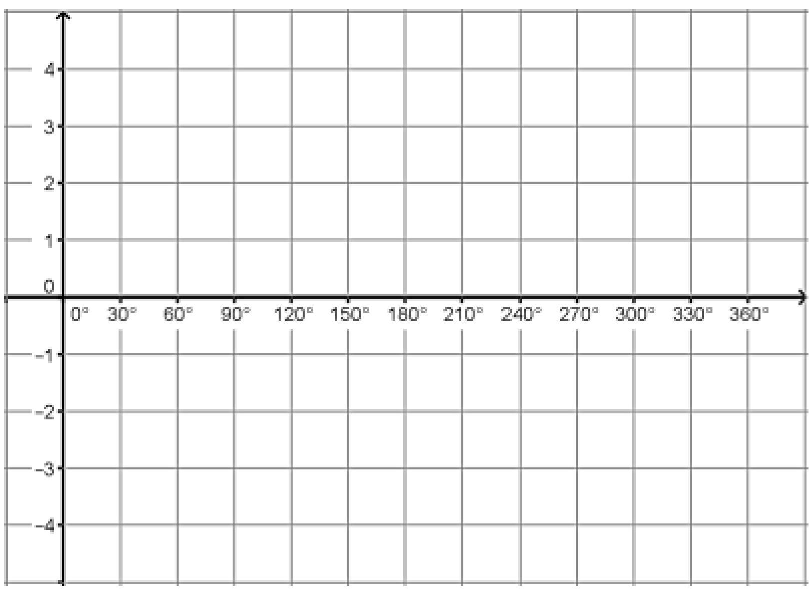
a. Explain how the graph would change if he built the Ferris wheel 2 meters above ground. Then write the new equation for the Ferris wheel.

New Equation:

b. Explain how the graph would change if he built the Ferris wheel 1 meter above the ground with a radius of 3. Then write the new equation for the Ferris wheel.

New Equation:

c. Graph both equations from part a and b on the grid below.



17. a. Explain how the graph would change if the Ferris wheel completes one rotation in half the time. Then write the new equation for the Ferris wheel.

New Equation:

b. Explain how the graph would change if it takes the Ferris wheel three times as long to complete one rotation. Then write the new equation for the Ferris wheel.

New Equation:

Practice

Determine the amplitude and midline of each trigonometric function. Then graph the function.

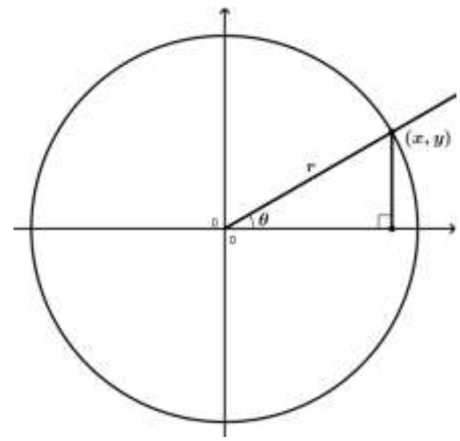
1. $y = 5\sin x - 1$ Amplitude______ Midline _________

2. $y = -3\sin x + 2$ Amplitude______ Midline _________

6.11 Warm Up
The Definition of Cosine

We begin with a circle of radius r whose center is located at the origin on a rectangular coordinate grid. We represent an angle of rotation in ***standard position*** by placing its vertex at the origin, the ***initial ray*** oriented along the positive x-axis, and its ***terminal ray*** rotated θ degrees counterclockwise around the origin when θ is positive and clockwise when θ is negative. Let the ordered pair (x, y) represent the point when the terminal ray intersects the circle. (See the diagram below)

In this diagram, angle θ is between 0 and 90°; therefore, the terminal ray is in quadrant I. A right triangle has been drawn in quadrant I similar to the right triangles we have drawn in the Ferris wheel tasks.



1. Based on this diagram and the right triangle definition of the cosine ratio, find an expression for $\cos\theta$ in terms of the variables *x, y* and *r.*

   $\cos\theta =$

Carlos and Clarita are making notes of what they have observed about this new way of defining $\cos\theta$.

Carlos: "For some angles the calculator gives me positive values for $\cos\theta$, and for other angles it gives me negative values."

2. Without using your calculator, list at least five angles of rotation for which the value of $\cos\theta$ produced by the calculator should be **positive**.

3. Without using your calculator, list at least five angles of rotation for which the value of $\cos\theta$ produced by the calculator should be **negative**.

6.11 Moving Shadows
A Practice Understanding Task

In spite of his nervousness, Carlos enjoys his first ride on the Ferris wheel. He does, however, spend much of his time with his eyes fixed on the ground below him. After a while, he becomes fascinated with the fact that, since the sun is directly overhead, his shadow moves back and forth across the ground beneath him as he rides around on the Ferris wheel.

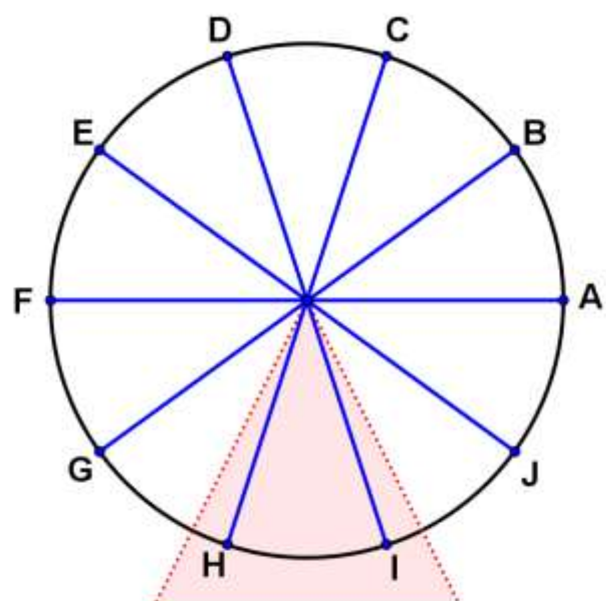
Recall the following facts for the Ferris wheel Carlos is riding:

- The Ferris wheel has a radius of 25 feet.
- The center of the Ferris wheel is 30 feet above the ground.
- The Ferris wheel makes one complete rotation counterclockwise every 20 seconds.

To describe the location of Carlos' shadow as it moves back and forth on the ground beneath him, we could measure the shadow's **horizontal distance** (in feet) to the right or left of the point directly beneath the center of the Ferris wheel, with locations to the right of the center having positive value and locations to the left of the center having negative values. For instance, in this system, Carlos' shadow's location will have a value of 25 when he is at the position farthest to the right of the center on the Ferris wheel, and a value of −25 when he is at a position farthest to the left of the center.

1. In this new measurement system, if Carlos' shadow is at 0 feet, where is he sitting on the Ferris wheel?

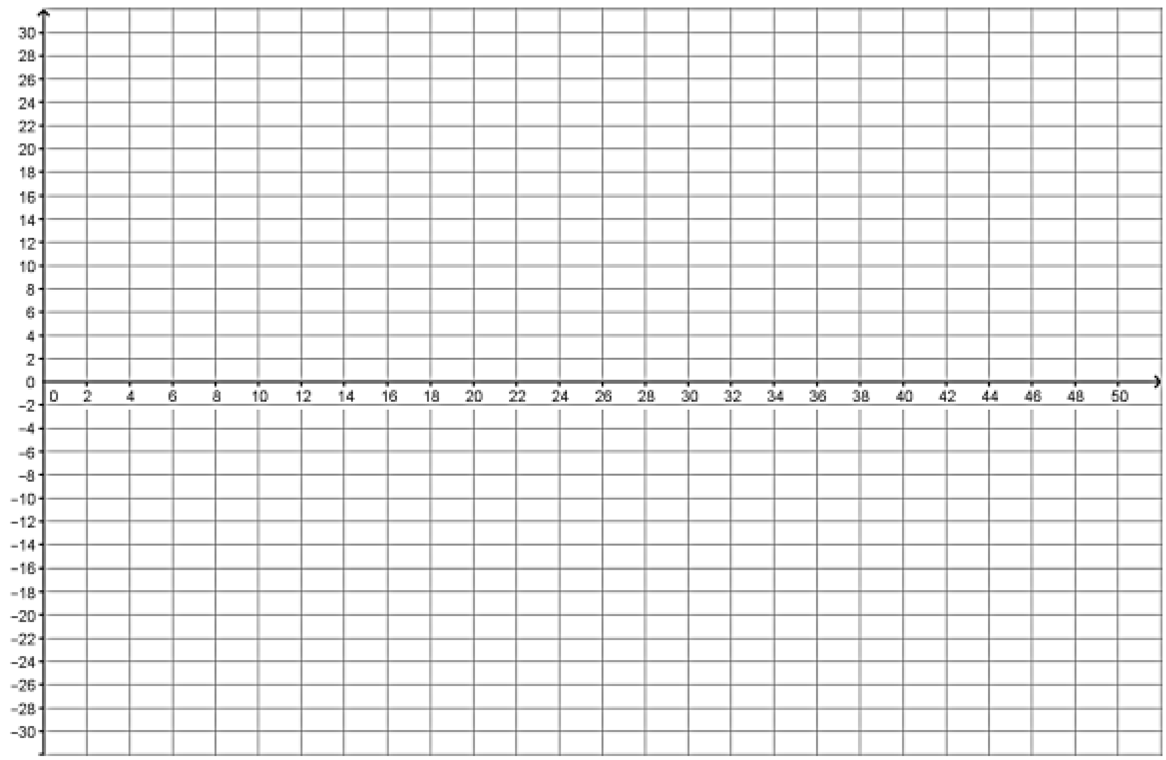
2. In our previous work with the Ferris wheel, t represents the number of seconds since Carlos passed the position farthest to the right on the Ferris wheel. How long will it take Carlos to be at the position described in question 1?



3. Calculate the location of Carlos' shadow from center at the times *t* given in the following table, where *t* represents the number of seconds since Carlos passed the position farthest to the right on the Ferris wheel. **Keep track of any patterns you notice in the ways you calculate the location of the shadow**. As you calculate each location, plot the location of the shadow from center on the graph on the following page.

Elapsed time since passing position A	Location of the shadow from center
1 sec	
3 sec	
5 sec	
7 sec	
8 sec	
10 sec	
12 sec	
13 sec	
14 sec	
16 sec	
17 sec	
18 sec	
20 sec	
22 sec	
23 sec	
25 sec	
27 sec	
28 sec	
29 sec	

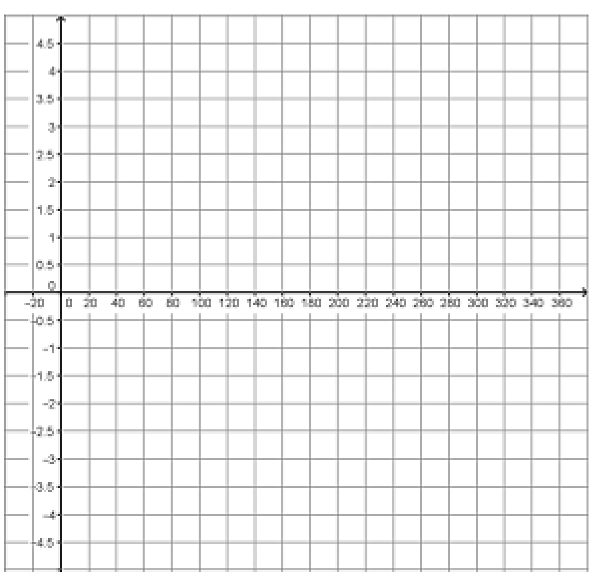
4. Sketch a graph of the horizontal location of Carlos' shadow from center as a function of time *t*, where *t* represents the elapsed time after Carlos passes position A, the farthest right position on the Ferris wheel.



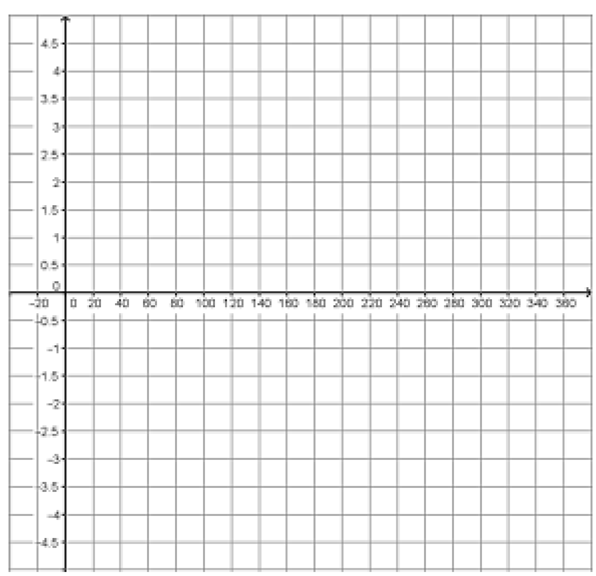
5. Write a general equation for finding the location of the shadow at any instant in time.

Graph the following functions.

6. $y = 2\cos x$

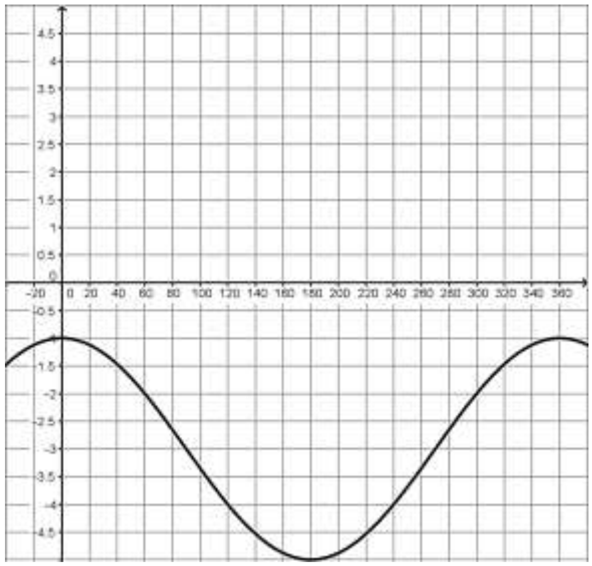


7. $y = -3\cos x + 2$

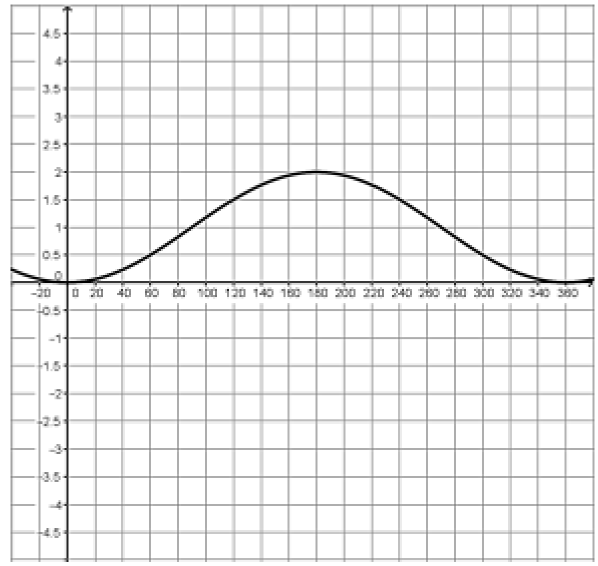


Given the graphs below, write at least one function that can be used to model the graph.

8.



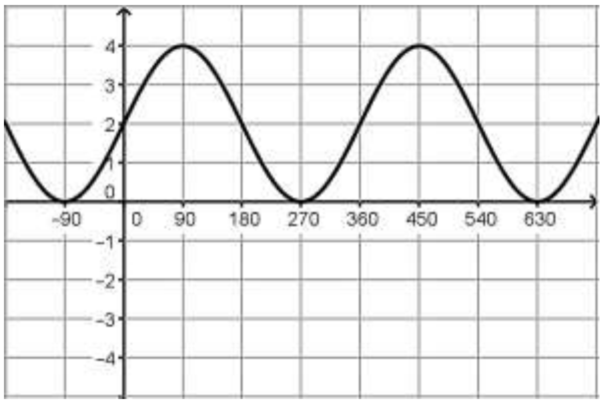
9.

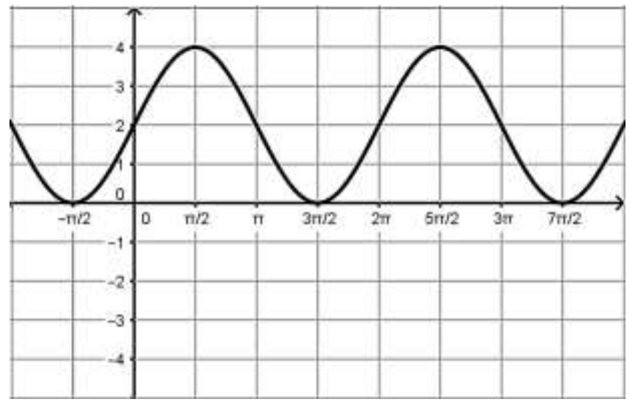


6.12 Warm Up
Transformations of Sine and Cosine

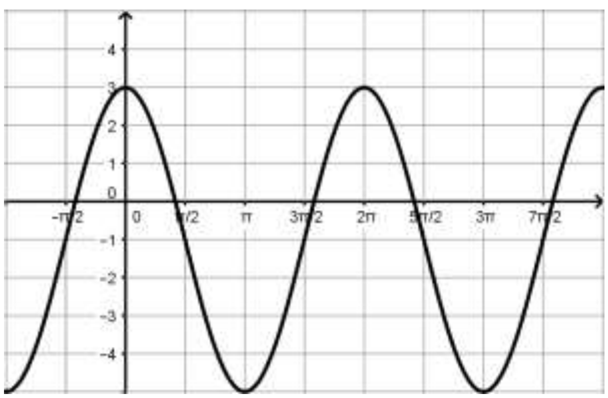
Find equations that describe the given graphs.

1.

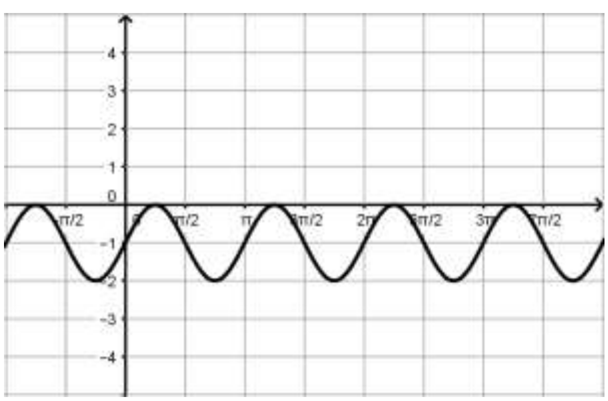




2.



3.



6.12 High Noon and Sunset Shadows
A Develop Understanding Task

In this task, we revisit the Ferris wheel that caused Carlos so much anxiety. Recall the following facts from previous tasks:

- The Ferris wheel has a radius of 25 feet.
- The center of the Ferris wheel is 30 feet above the ground.
- The Ferris wheel makes one complete rotation counterclockwise every 20 seconds.

The Ferris wheel is located next to a high-rise office complex. At sunset, a rider casts a shadow on the exterior wall of the high-rise building. As the Ferris wheel turns, you can watch the shadow of the rider rise and fall along the surface of the building. In fact, you know an equation that would describe the height of this "sunset shadow."

1. Write the equation of this "sunset shadow."

At noon, when the sun is directly overhead, a rider casts a shadow that moves left and right along the ground as the Ferris wheel turns. In fact, you know an equation that would describe the motion of this "high noon shadow."

2. Write the equation of this "high noon shadow."

3. Based on your previous work, you probably wrote these equations in terms of the angle of rotation being measured in degrees. Revise your equations so the angle of rotation is measured in **radians**.

   a. The "sunset shadow" equation in terms of radians:

   b. The "high noon shadow" equation in terms of radians:

4. In the equations you wrote in question 3, where on the Ferris wheel was the rider located at time $t = 0$? We will refer to the position as the rider's ***initial position*** on the wheel.

   a. Initial position for the "sunset" shadow equation:

   b. Initial position for the "high noon" shadow equation:

5. Revise your equations from question 3 so that the rider's initial position at $t = 0$ is **at the top of the wheel**.

   a. The "sunset shadow" equation, initial position at the top of the wheel:

   b. The "high noon shadow" equation, initial position at the top of the wheel:

6. Revise your equations from question 3 so that the rider's initial position at $t = 0$ is **at the bottom of the wheel**.

   a. The "sunset shadow" equation, initial position at the bottom of the wheel:

   b. The "high noon shadow" equation, initial position at the bottom of the wheel:

7. Revise your equations from question 3 so that the rider's initial position at $t = 0$ is **at the point farthest to the left of the wheel.**

   a. The "sunset shadow" equation, initial position at the point farthest to the left of the wheel:

   b. The "high noon shadow" equation, initial position at the point farthest to the left of the wheel:

8. Revise your equations from question 3 so that the rider's initial position at $t = 0$ is **halfway between the farthest point to the right on the wheel and the top of the wheel.**

   a. The "sunset shadow" equation, initial position halfway between the farthest point to the right on the wheel and the top of the wheel:

   b. The "high noon shadow" equation, initial position halfway between the farthest point to the right on the wheel and the top of the wheel:

9. Revise your equations from question 3 so that the wheel **rotates twice as fast**.

   a. The "sunset shadow" equation for the wheel rotating twice as fast:

   b. The "high noon shadow" equation for the wheel rotating twice as fast:

10. Revise your equations from question 3 so that the **radius of the wheel is twice as large and the center of the wheel is twice as high.**

    a. The "sunset shadow" equation for a radius twice as large and the center twice as high:

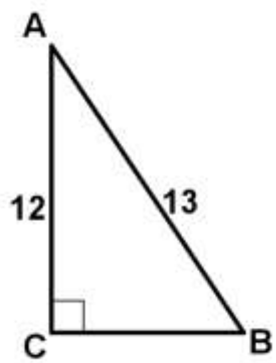
    b. The "high noon shadow" equation for a radius twice as large and the center twice as high:

6.13 Warm Up
What's Your Angle?

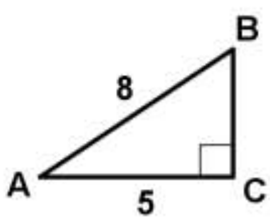
For each triangle:

- Write an equation that includes a trigonometric function.
- Solve your equation to find the measurements of each acute angle. Round all answers to the nearest whole number.

1.



2.



6.13 High Tide
A Solidify Understanding Task

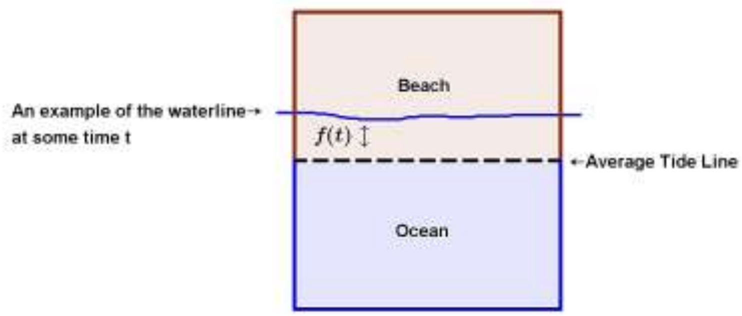
Perhaps you have built an elaborate sand castle at the beach only to have it get swept away by the in-coming tide.

Frustrated by having your sandcastle knocked down by the tide, you decide to pay attention to the tides so that you can keep track of how much time you have to build and admire your sand castle.

You have a friend who is in calculus who will be going to the beach with you. You give your friend some data from the almanac about high tides along the ocean, as well as a contour map of the beach you intend to visit, and ask her to come up with an equation for the water level on the beach on the day of your trip. According to your friend's analysis, the water level on the beach will fit this equation:

$$f(t) = 20\sin\left(\frac{\pi}{6}t\right)$$

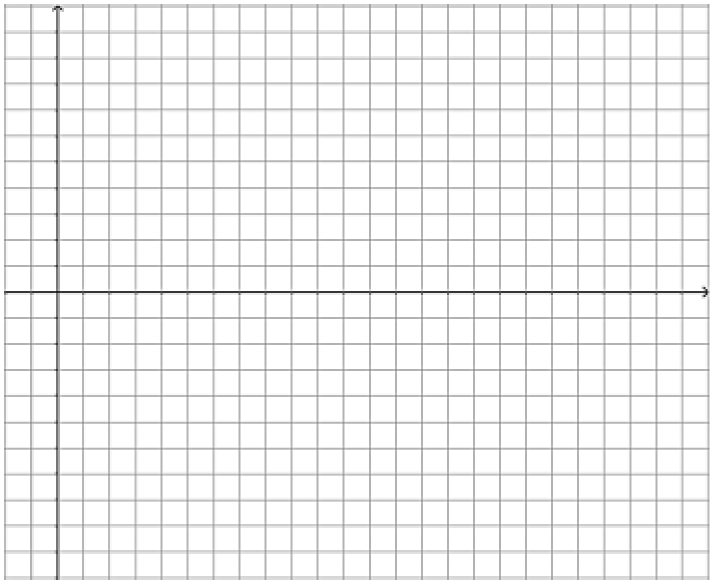
In this equation, $f(t)$ represents how far the tide is "in" or "out" from the average tide line. This distance is measured in feet and t represents the elapsed time, in hours, since midnight.



1. What is the highest up the beach (compared to its average position) that the tide will be during the day? This is called ***high tide***. What is the lowest that the tide will be during the day? This is called ***low tide***.

2. Suppose you plan to build your castle right on the average tide line just as the water has moved below that line. How much time will you have to build your castle before the incoming tide destroys your work?

3. Use the given function and your answers to questions 1 and 2 to sketch the graph of two complete tide cycles. Label and scale the axes.



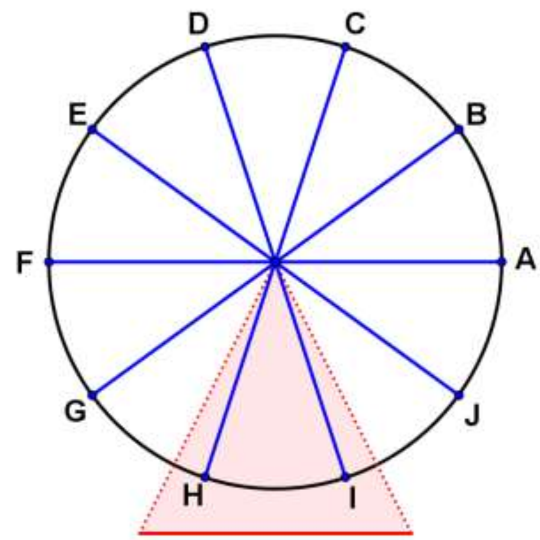
4. Suppose you want to build your castle 10 feet below the average tide line to take advantage of the damp sand. What is the maximum amount of time you will have to make your castle? How can you convince your friend that your answer is correct? Use your graph to estimate this amount of time and then use algebra & the inverse sine function to find the exact answer.

5. Suppose you want to build your castle 15 feet above the average tide line to give you more time to admire your work. What is the maximum amount of time you will have to make your castle? How can you convince your friend that your answer is correct? Use your graph to estimate this amount of time and then use algebra & the inverse sine function to find the exact answer.

6. Suppose you decide you only need two hours to build and admire your castle. What is the lowest point on the beach where you can build the castle?

6.14 Getting on the Right Wavelength
A Practice Understanding Task

Below is a new Ferris wheel that has a radius of 40 feet, whose center is 50 feet from the ground, and makes one revolution counterclockwise every 18 seconds.



1. Write the equation of the height from the ground of the rider at any time t, if at $t = 0$ the rider is at **position A**. Use radians to measure the angles of rotation.

2. At what time(s) is the rider 70 feet above the ground? Show the details of how you answered this question.

3. Use your answer from question 2 to write an equation to show when the rider is 70 feet above the ground if the Ferris wheel goes around forever.

4. If you used a sine function in question 1, revise your equation to model the same motion with a cosine function. If you used a cosine function in question 1, revise your equation to model the motion with a sine function.

5. Write the equation of the height of the rider at any time t, if at $t = 0$, the rider is at **position D**. Use radians to measure the angles of rotation.

6. For the equation you wrote in question 4, at what time(s) is the rider 80 feet above the ground? Show or explain the details of how you answered this question.

7. Use your answer from question 6 to write an equation to show when the rider is 80 feet above the ground if the Ferris wheel goes around forever.

8. If you used a sine function in question 4, revise your equation to model the same motion with a cosine function. If you used a cosine function, revise your equation to model the motion with a sine function.

9. Choose any other position on the Ferris wheel as the starting position at $t = 0$. Write the equation of the height of the rider at any time t. Use radians to measure the angles of rotation. Also change at least one other feature of the Ferris wheel such as the height of the center, the radius, the direction of rotation and/or the length of time for a single rotation. Record your equation and description of your Ferris wheel here.

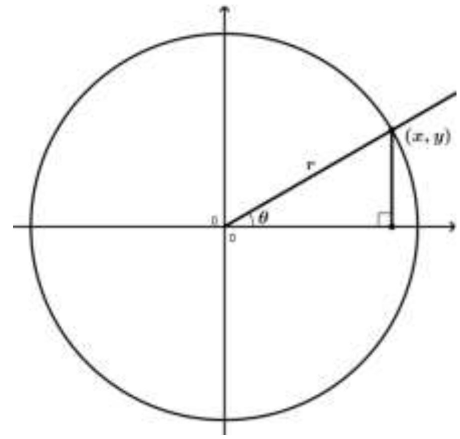
10. Trade the equation you wrote in question 7 with a partner and see if he or she can determine the essential features of your Ferris wheel: height of center, radius, period, direction of rotation, starting position of the rider. Discuss any issues where you and your partner have differences in your descriptions of the Ferris wheel modeled by your equation.

6.15 Off on a Tangent
A Develop and Solidify Understanding Task

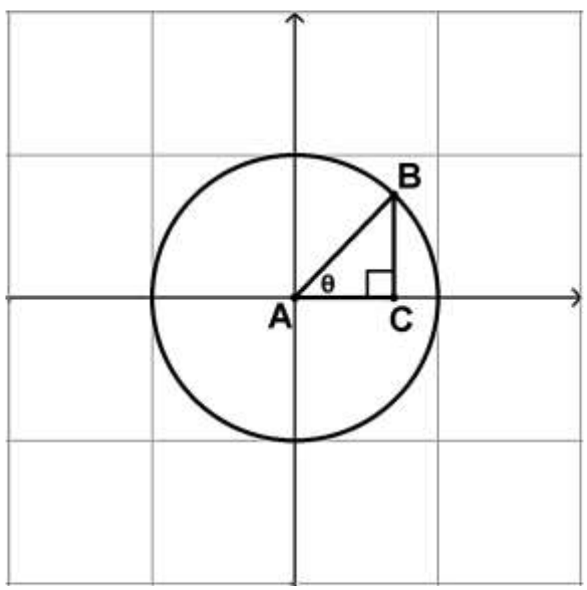
Recall that the right triangle definition of the tangent ratio is:

$$\tan\theta = \frac{\text{length of side opposite angle } \theta}{\text{length of side adjacent to angle } \theta}$$

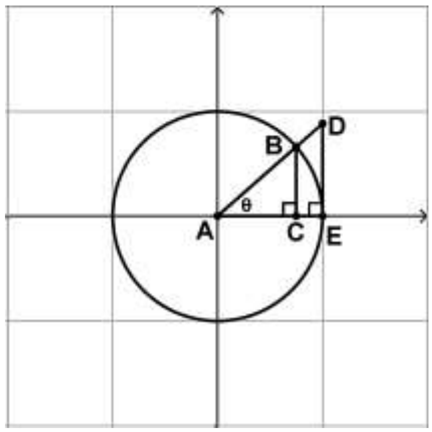
1. Revise the right triangle definition of tangent to find the tangent of **any angle of rotation** drawn in standard position on the unit circle. Explain why your definition is reasonable.



2. We have observed that on the unit circle the value of sine and cosine can be represented with the length of a line segment. Indicate on the following diagram which segment's length represents the value of $\sin\theta$ and which represents the value of $\cos\theta$ for the given angle θ.



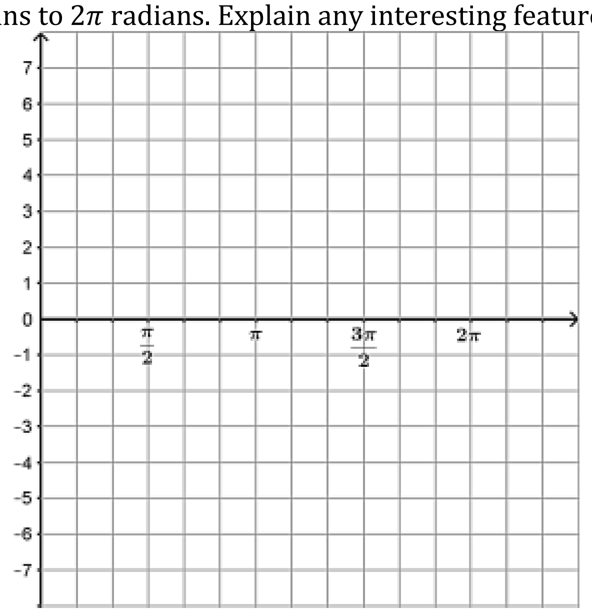
3. There is also a line segment that can be defined on the unit circle so that its length represents the value of $\tan\theta$. Consider the length of $\overline{DE}$ in the unit circle diagram below. Note that ΔADE and ΔABC are similar right triangles. Write a convincing argument explaining why the length of $\overline{DE}$ is equivalent to the value of $\tan\theta$ for the given angle, θ.



4. Extend your thinking about $y = \tan\theta$ by considering the length of $\overline{DE}$ as θ rotates through positive angles from 0 radians to 2π radians. Using your unit circle diagrams from the task, *Water Wheels and the Unit Circle*, give exact values for the following trigonometric expressions:

a. $\tan\frac{\pi}{6} =$	b. $\tan\frac{5\pi}{6} =$	c. $\tan\frac{7\pi}{6} =$
d. $\tan\frac{\pi}{4} =$	e. $\tan\frac{3\pi}{4} =$	f. $\tan\frac{11\pi}{6} =$
g. $\tan\frac{\pi}{2} =$	h. $\tan\pi =$	i. $\tan\frac{7\pi}{3} =$
j. $\tan -\frac{\pi}{3} =$	k. $\tan\frac{3\pi}{2} =$	l. $\tan -\frac{\pi}{4} =$

5. On the coordinate axes below, sketch the graph of $y = \tan\theta$ by considering the length of $\overline{DE}$ as θ rotates through angles from 0 radians to 2π radians. Explain any interesting features you notice in your graph.



6. Based on these definitions and your work in this module, determine how to classify each of the following trigonometric functions.

- A function $f(x)$ is classified as an **odd function** if $f(-\theta) = -f(\theta)$.
- A function $f(x)$ is classified as an **even function** if $f(-\theta) = f(\theta)$.

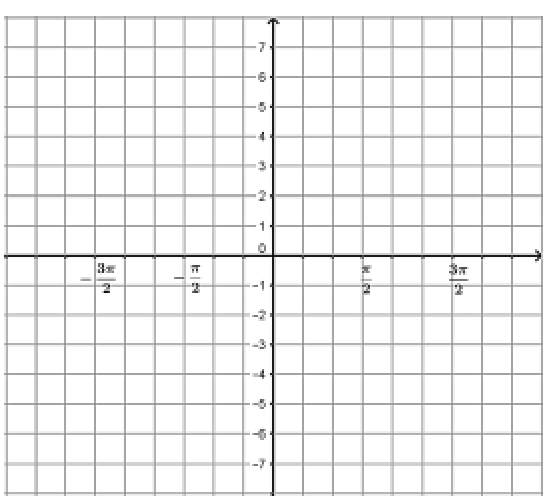
a. The function $y = \sin x$ would be classified as an [**odd function, even function, neither an even or odd function**]. Give evidence for your response.

b. The function $y = \cos x$ would be classified as an [**odd function, even function, neither an even or odd function**]. Give evidence for your response.

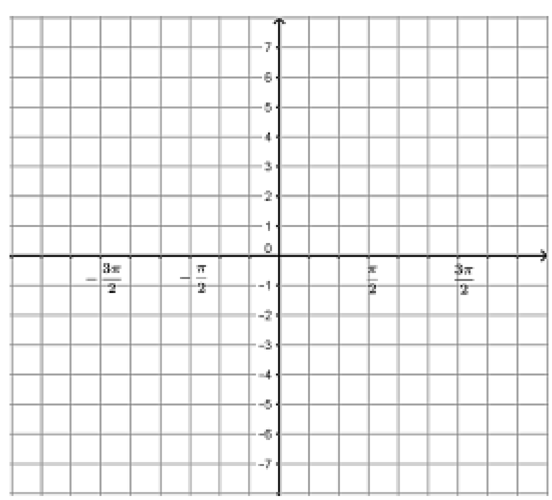
c. The function $y = \tan x$ would be classified as an [**odd function, even function, neither an even or odd function**]. Give evidence for your response.

7. Graph each tangent function. Be sure to identify the locations of the asymptotes.

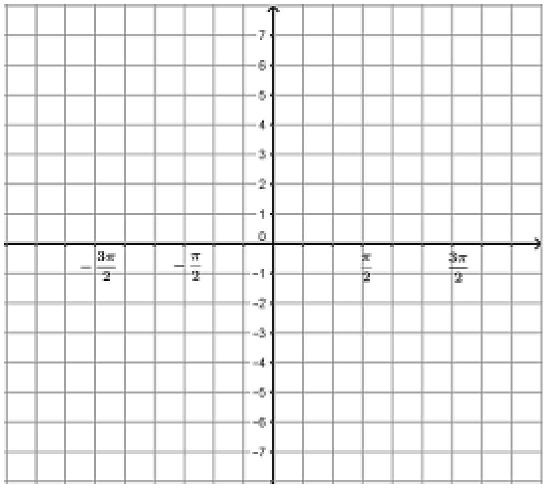
a. $y = \tan 2\theta$



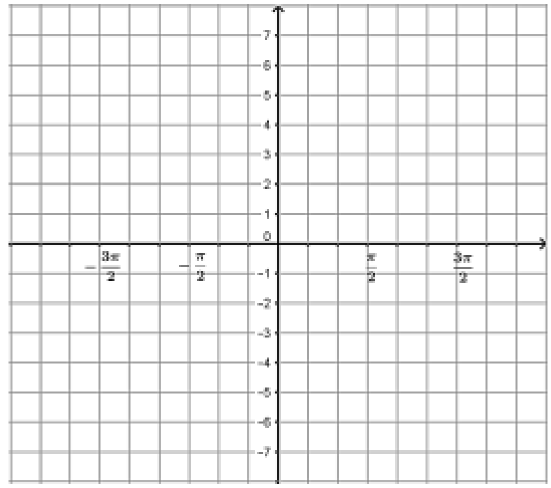
b. $y = 2\tan\theta - 3$



c. $y = 1 + \tan\frac{1}{2}\theta$

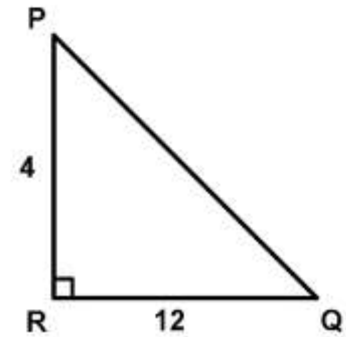


d. $y = -\tan\theta + 2$



6.16 Warm Up
Reciprocal Trigonometric Values

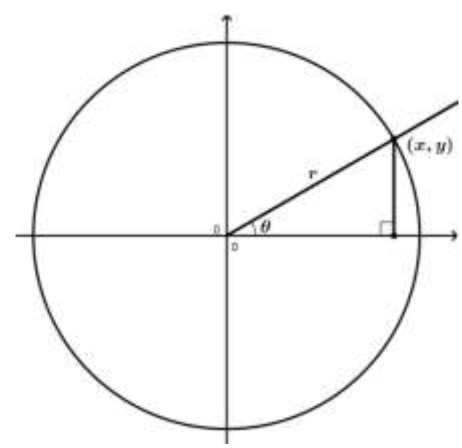
Use the triangle below to identify the following trigonometric ratios. Be sure to give exact values.



1. $\sin Q =$
2. $\csc Q =$
3. $\cos Q =$
4. $\sec Q =$
5. $\tan Q =$
6. $\cot Q =$

Recall how we extended the definitions of sine, cosine, and tangent for all angles of rotation:

$\boldsymbol{\sin\theta = \frac{y}{r}}$ $\boldsymbol{\cos\theta = \frac{x}{r}}$ $\boldsymbol{\tan\theta = \frac{y}{x}}$



7. How might the definitions of cosecant, secant, and cotangent be extended to all angles of rotation?

   $\csc\theta =$

   $\sec\theta =$

   $\cot\theta =$

6.16 Reciprocating the Graphs
A Develop and Solidify Understanding Task

Carlos and Clarita were graphing trigonometric functions on their math homework and were wondering what the graphs of the reciprocal functions would look like.

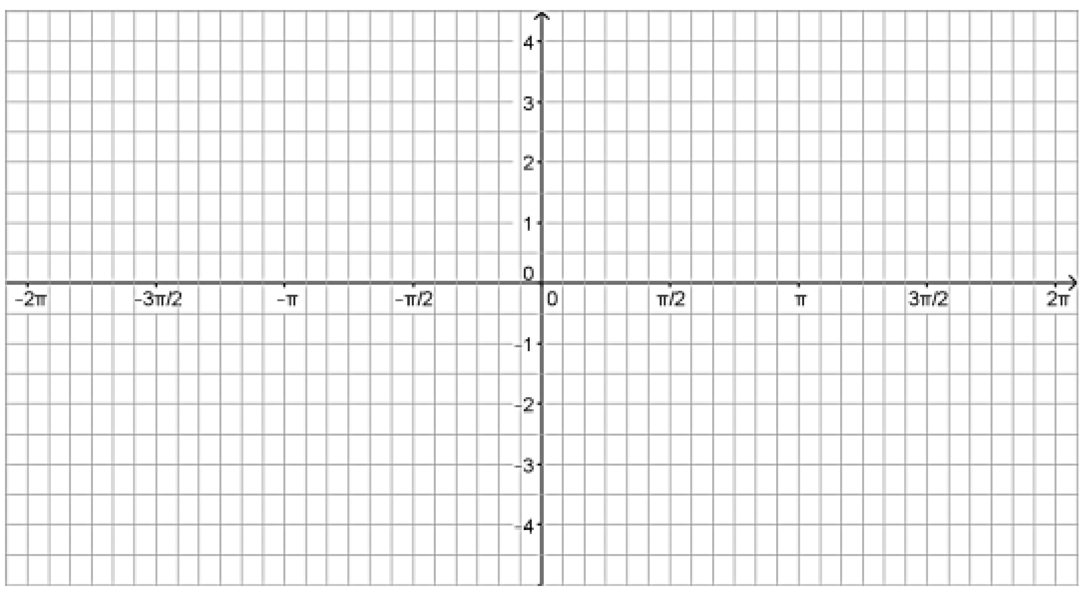
Carlos stated: "Since $\sin\theta$ and $\cos\theta$ both have maximum and minimum values of 1 and -1, I think these will be the maximum and minimum values for $\csc\theta$ and $\sec\theta$ as well."

Clarita disagreed: "No way! Think about the values between the zeros and the maximum or between the zeros and the minimum. What happens if you take their reciprocals? The numbers become really large."

1. Who do you agree with? Explain.

2. Complete the table of values for angles in quadrant I for $y = \sin\theta$ and $y = \csc\theta$. Graph these two functions in different colors on the coordinate plane. Use symmetry to complete the sketch of each function for angles on a domain of $[-2\pi, 2\pi]$. Reminder: $\csc\theta = \frac{1}{\sin\theta}$.

θ	$\sin\theta$	$\csc\theta$
0		
$\frac{\pi}{6}$		
$\frac{\pi}{4}$		
$\frac{\pi}{3}$		
$\frac{\pi}{2}$		

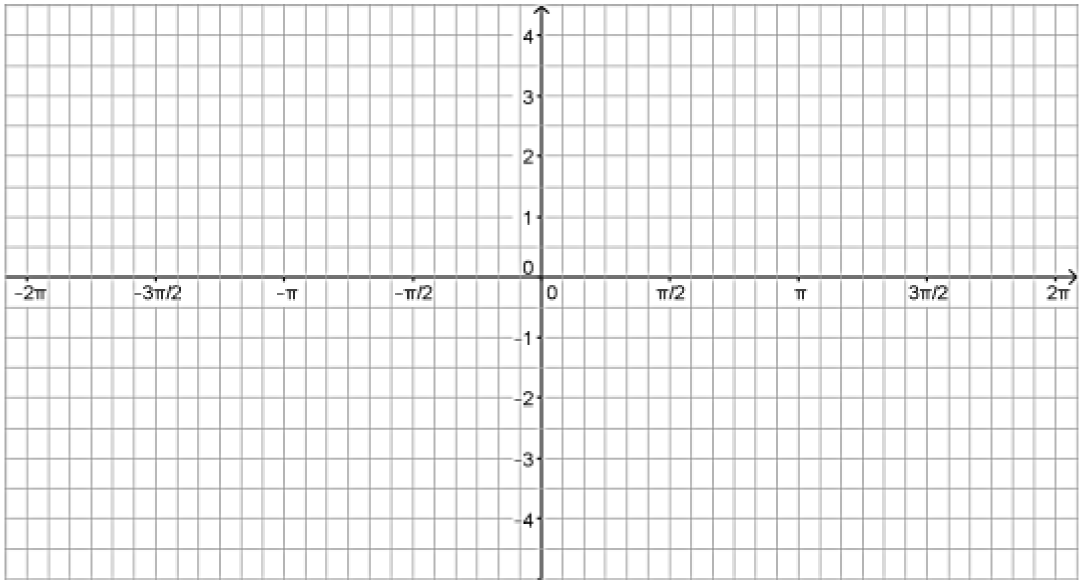


3. What happens to the graph of $y = \csc\theta$ when the graph of $y = \sin\theta$ crosses the x-axis?

4. Explain why $y = \csc\theta$ has the shape that appears in your graph in question 2.

5. Complete the table of values for angles in quadrant I for $y = \cos\theta$ and $y = \sec\theta$. Graph these two functions in different colors on the coordinate plane. Use symmetry to complete the sketch of each function for angles on a domain of $[-2\pi, 2\pi]$. Reminder: $\sec\theta = \frac{1}{\cos\theta}$.

θ	$\cos\theta$	$\sec\theta$
0		
$\frac{\pi}{6}$		
$\frac{\pi}{4}$		
$\frac{\pi}{3}$		
$\frac{\pi}{2}$		



6. What happens to the graph of $y = \sec\theta$ when the graph of $y = \cos\theta$ crosses the x-axis?

7. Explain why $y = \sec\theta$ has the shape that appears in your graph in question 5.

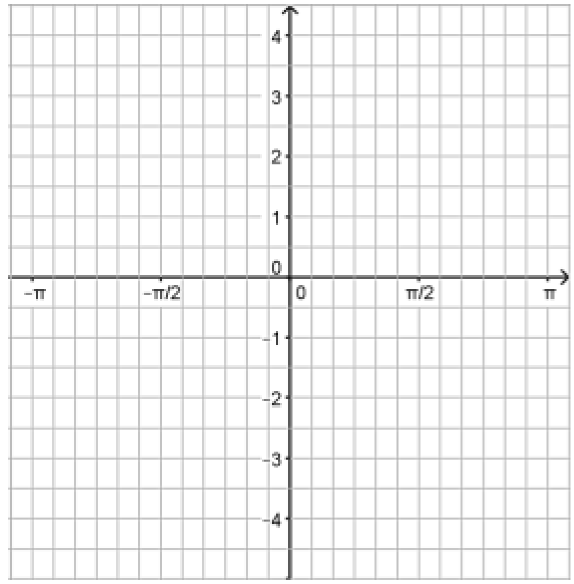
Clarita says: "The graphs of $y = \csc\theta$ and $y = \sec\theta$ look so strange. What would happen if we take the reciprocals of the values of $\tan\theta$?"

Carlos adds: "That graph will look crazy because $y = \tan\theta$ has asymptotes. What is the reciprocal of an asymptote?"

8. Answer Carlos' question: What is the reciprocal of an asymptote? Hint: think about what type of ratio creates an asymptote.

9. Complete the table of values for angles in quadrant I for $y = \tan\theta$ and $y = \cot\theta$. Graph these two functions in different colors on the coordinate plane. Use symmetry to complete the sketch of each function for angles on a domain of $[-\pi, \pi]$. Reminder: $\cot\theta = \frac{1}{\tan\theta}$.

θ	$\tan\theta$	$\cot\theta$
0		
$\frac{\pi}{6}$		
$\frac{\pi}{4}$		
$\frac{\pi}{3}$		
$\frac{\pi}{2}$		



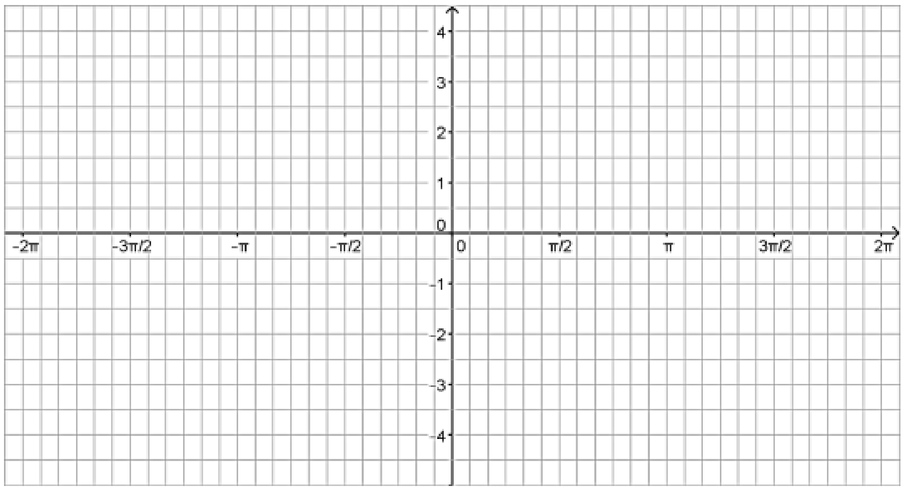
10. What happens to the graph of $y = \cot\theta$ when the graph of $y = \tan\theta$ crosses the x-axis?

11. What happens to the graph of $y = \cot\theta$ when the graph of $y = \tan\theta$ has an asymptote?

Carlos: “Now that we have seen the basic graphs of $y = \csc\theta$, $y = \sec\theta$, & $y = \cot\theta$, what would happen if we change the numbers in the functions?”

Clarita responds: “Okay. Let’s try graphing $y = 2\csc\theta$ by first graphing $y = 2\sin\theta$.”

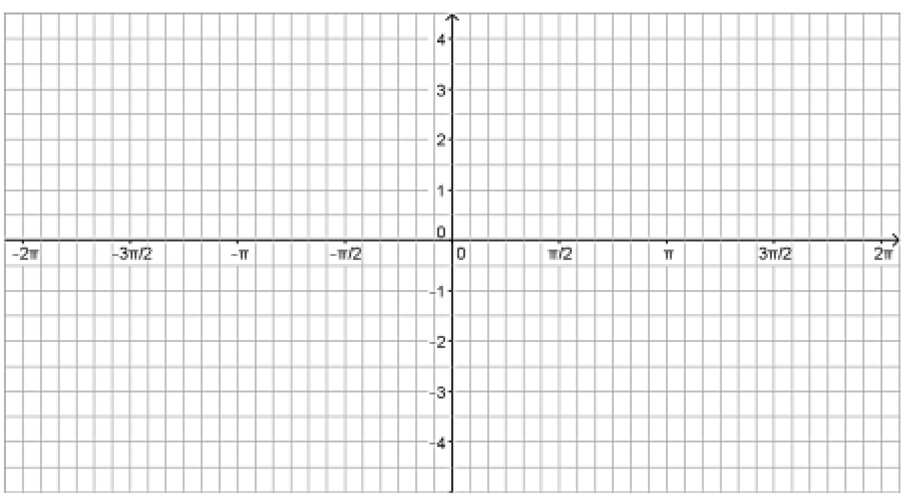
12. Graph the function $y = 2\csc\theta$ using Clarita’s recommendation.



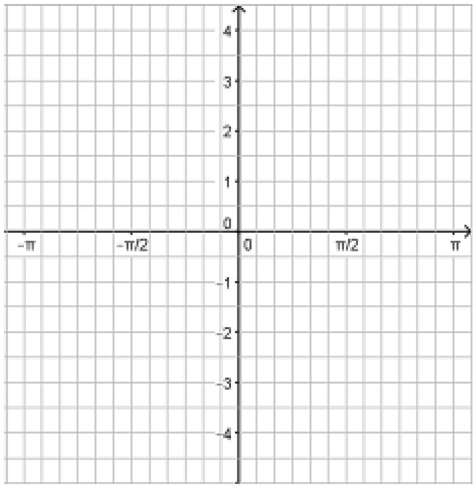
Carlos says: “I’m starting to understand. Let’s try a few more.”

Graph the following functions by first graphing the associated sine, cosine, or tangent function.

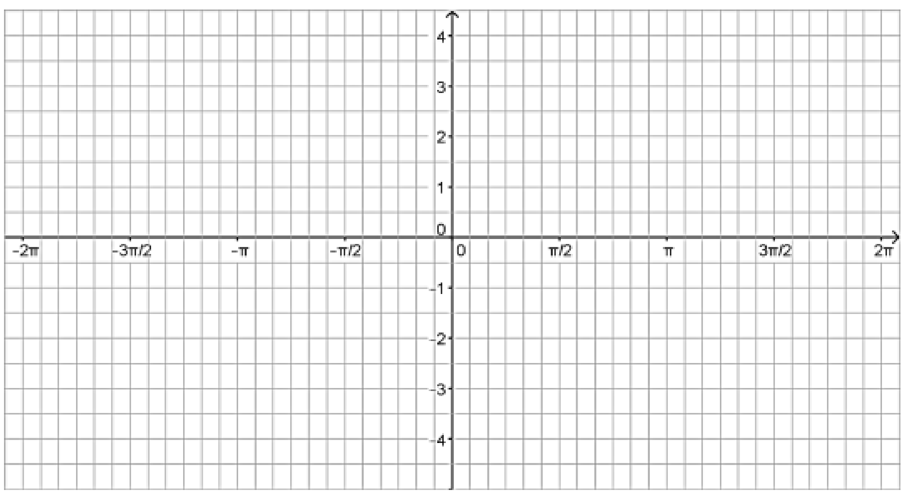
13. $y = \sec 2\theta$



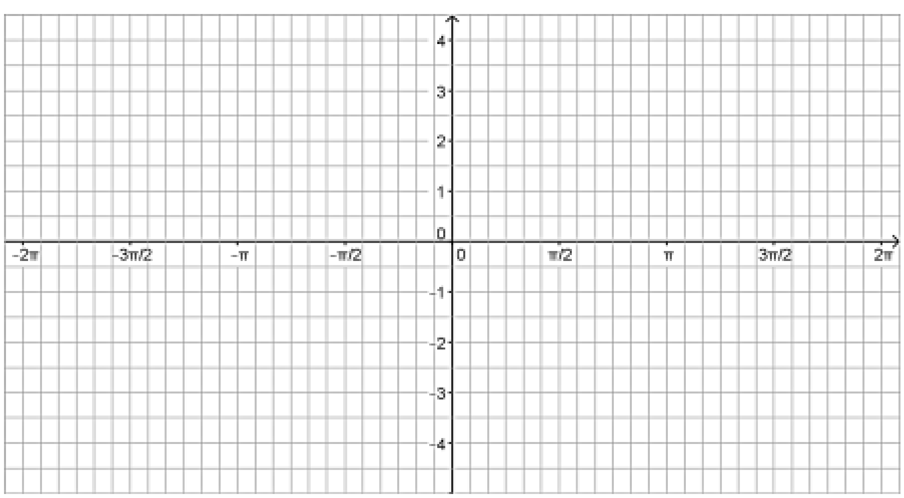
14. $y = 3\cot\theta$



15. $y = \sec\theta - 2$



16. $y = \csc\theta + 1$



Integrated Math 3
Module 6
Trigonometric Functions
Ready, Set, Go! Homework

Adapted from

The Mathematics Vision Project:
Scott Hendrickson, Joleigh Honey, Barbara Kuehl,
Travis Lemon, Janet Sutorius

Name: **Trigonometric Functions | 6.1**

Ready, Set, Go!

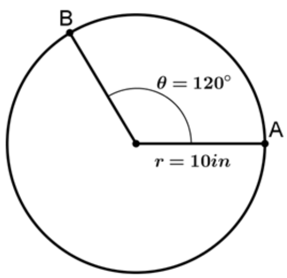
Ready

Topic: Finding the length of an arc using proportions

Use the given degree measure of the central angle to set up a proportion to find the length of minor arc AB or the length of the semicircle. Leave your answers in terms of π.

Recall that $s = \frac{\theta}{360°}(\pi d)$ where s is the arc length.

1.



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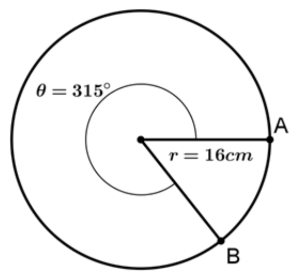

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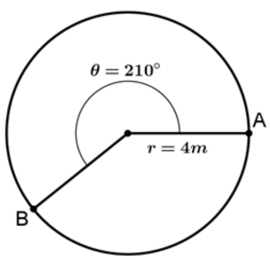

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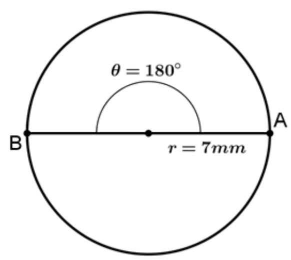

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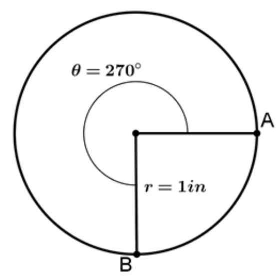
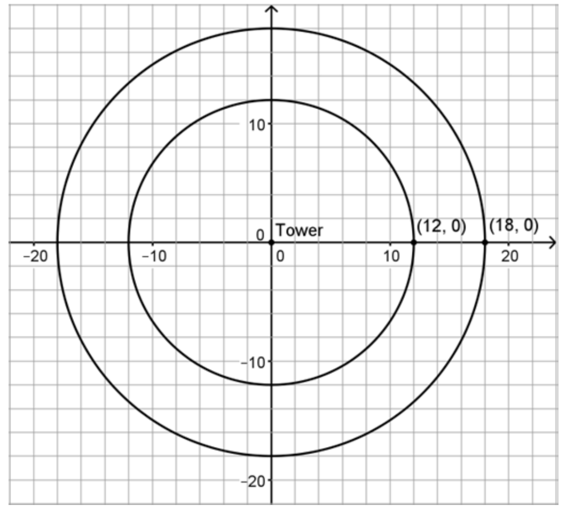


6. The circumference of circle A is 400 meters. The circumference of circle B is 800 meters. What is the relationship between the radius of circle A and the radius of circle B? Justify your answer.

Set

Topic: Measuring central angles

Veronica now thinks that the model of the archaeological site needs to have stakes that are equidistant from the vertical and horizontal axes. Therefore, she proposes using **8 evenly spaced stakes** around each circle. Alyce also wants to make sure they record the distance around the circle to each new stake. As before, the central tower is located at the origin and the first of each set of 8 stakes for the inner and outer circles is placed at the points $(12, 0)$ and $(18, 0)$, respectively.



7. Your job is to determine the x- and y-coordinates for each of the remaining 8 stakes on each circle, as well as the arc lengths from the points $(12, 0)$ or $(18, 0)$, depending on which circle the stake is located. Keep track of the solutions in the table below.

Javier suggests they record the location of each stake and its distance around the circle for the set of stakes on each circle. Veronica suggests it might also be interesting to record the ratio of the arc length to the radius for each circle.

8. Help Javier and Veronica complete this table.

	Inner Circle: r = 12 meters			**Outer Circle: r = 18 meters**		
	Location	**Distance from (12, 0) along circular path**	**Ratio of arc length to radius**	**Location**	**Distance from (18, 0) along circular path**	**Ratio of arc length to radius**
Stake 1	$(12, 0)$	0	0	$(18, 0)$	0	0
Stake 2						
Stake 3						
Stake 4						
Stake 5						
Stake 6						
Stake 7						
Stake 8						

9. What interesting patterns might Alyce, Javier, and Veronica notice in their work and their table? Summarize any interesting things you have noticed.

Go

Topic: Converting angles between radians and degrees

Recall that there are **360 degrees** in a full circle and **2π radians** in a full circle. Thus, $360° = 2\pi$ radians. If we divide both sides of the equation by 2, we create another identity $180° = \pi$ radians.

One method to convert degrees to radians or radians to degrees uses this identity:

Since $180° = \pi$ radians, it follows that $\frac{\pi}{180°} = \frac{180°}{\pi} = 1$

If I want to convert 72° into radian measure, then I need the unit of degrees to cancel, so I will multiply 72° by $\frac{\pi}{180°}$, $72° \cdot \frac{\pi}{180°} = \frac{72° \times \pi}{180°} = \frac{\pi}{5}$. Hence, an angle that measures 72° is equivalent to $\frac{\pi}{5}$.

Another method to convert degrees to radians or radians to degrees is to set up a proportion:

Since the ratio of degrees to radians is 180° to π radians, it follows that 72° to r radians is proportional to 180° to π radians, or $\frac{72°}{r} = \frac{180°}{\pi}$

Convert the following angles from degrees to radians or radians to degrees.

10. 45°

11. 15°

12. 54°

13. 135°

14. 300°

15. 270°

16. $\frac{5\pi}{6}$

17. $\frac{\pi}{8}$

18. $\frac{3\pi}{4}$

19. $\frac{7\pi}{5}$

20. 2 radians

21. 15 radians

Name: **Trigonometric Functions | 6.2**

Ready, Set, Go!

Ready

Topic: Finding points on a circle

Given the equation of a circle centered at $(0, 0)$, find one point in each quadrant that lies on the given circle.

1. $x^2 + y^2 = 25$ quadrant I: quadrant II:
   quadrant III: quadrant IV:

2. $x^2 + y^2 = 169$ quadrant I: quadrant II:
   quadrant III: quadrant IV:

3. $x^2 + y^2 = 36$ quadrant I: quadrant II:
   quadrant III: quadrant IV:

4. $x^2 + y^2 = 1$ quadrant I: quadrant II:
   quadrant III: quadrant IV:

5. $x^2 + y^2 = 9$ quadrant I: quadrant II:
   quadrant III: quadrant IV:

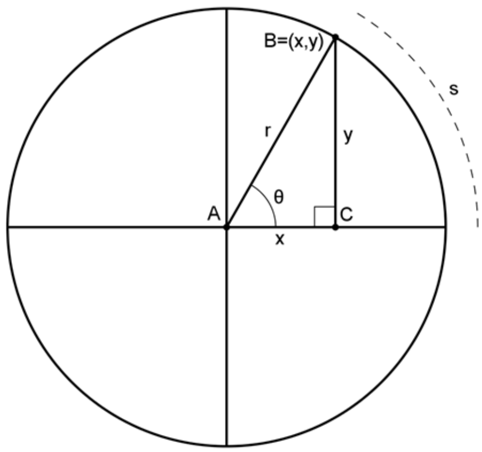
6. $x^2 + y^2 = 4$ quadrant I: quadrant II:
   quadrant III: quadrant IV:

Set

Topic: Locating points in terms of rectangular coordinates, arc length, reference angle, and radius

In the diagram, ΔABC is a right triangle. Point B lies on the circle and is described by the rectangular coordinates (x, y), s is the length of the intercepted arc created by angle θ, r is the radius of circle A.

Answer the following questions using the given information.



7. B has rectangular coordinates $(5, 12)$.
   a. Find r.

   b. Find θ to the nearest tenth of a degree.

   c. Find s by using the formula $s = \frac{\theta}{360°}(d\pi)$.

   d. Describe point B using the coordinates (r, θ).

   e. Describe point B using the radius and arc length (r, s).

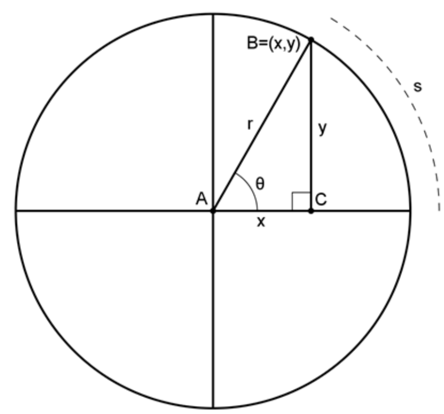
8. B has rectangular coordinates $(33, 56)$.
   a. Find r.

   b. Find θ to the nearest tenth of a degree.

   c. Find s by using the formula $s = \frac{\theta}{360°}(d\pi)$.

   d. Describe point B using the coordinates (r, θ).

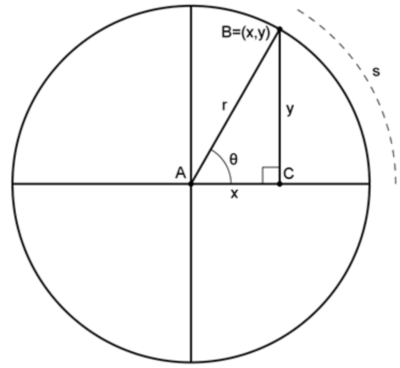
   e. Describe point B using the radius and arc length (r, s).



9. B is described by (r, θ) where $\theta \approx 58.11°$ and $r = 53$.
   a. Find (x, y) to the nearest whole number

   b. Find s by using the formula $s = \frac{\theta}{360°}(d\pi)$.

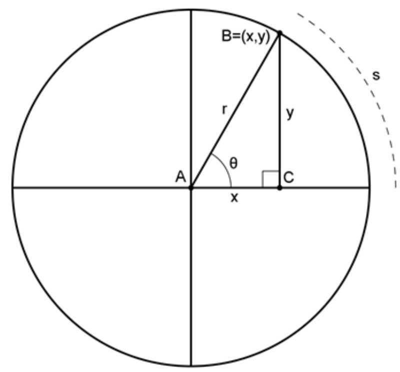
   c. Describe point B using the radius and arc length (r, s).



10. B is described by (r, θ) where $\theta \approx 25.01°$ and $r = 85$.
   a. Find (x, y) to the nearest whole number

   b. Find s by using the formula $s = \frac{\theta}{360°}(d\pi)$.

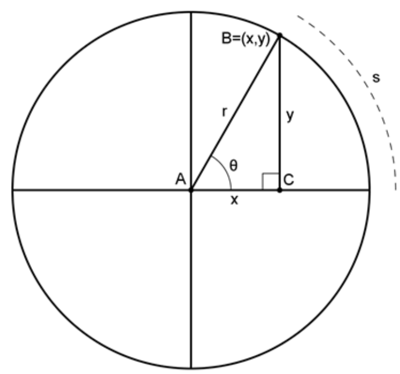
   c. Describe point B using the radius and arc length (r, s).



11. B is described by (r, s) where $s \approx 46$ and $r = 37$.
   a. Find θ by using the formula $s = \frac{\theta}{360°}(d\pi)$.

   b. Find (x, y) to the nearest whole number

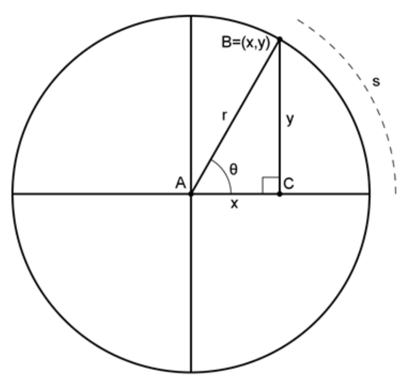
   c. Describe point B using the coordinates (r, θ).



12. B is described by (r, s) where $s \approx 62.26$ and $r = 73$.
   a. Find θ by using the formula $s = \frac{\theta}{360°}(d\pi)$.

   b. Find (x, y) to the nearest whole number

   c. Describe point B using the coordinates (r, θ).



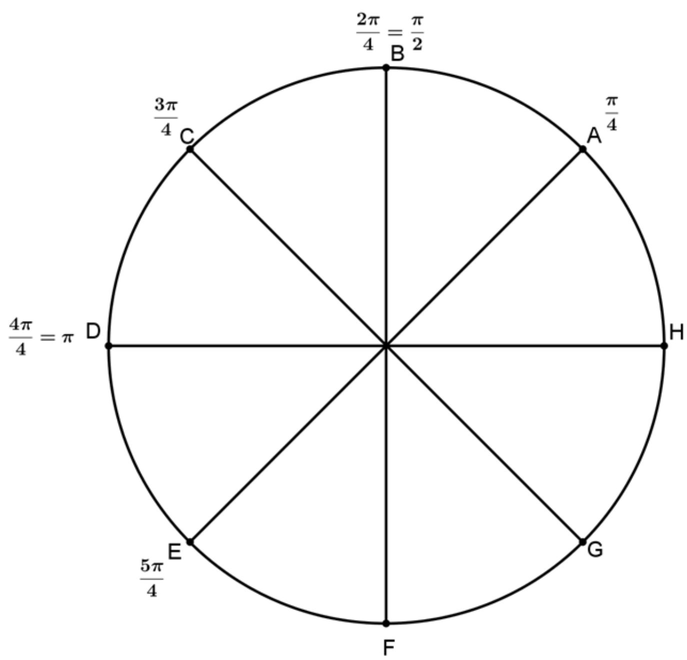
Go

Topic: Radian measurement

Label each point on the circle with the measure of the angle of rotation in standard position. Angle measures should be in <u>radians</u>. (Recall that a full rotation around the circle would be 2π radians.)

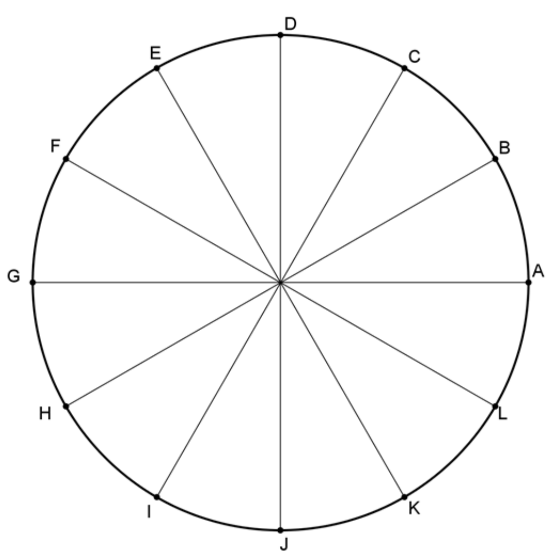
Example: The circle has been divided equally into 8 parts. Each part is equal to $\frac{2\pi}{8}$ or $\frac{\pi}{4}$ radians. Indicate how many parts of $\frac{\pi}{4}$ radians there are at each position around the circle.

13. Finish the example by writing the angle measures for points F, G, and H.

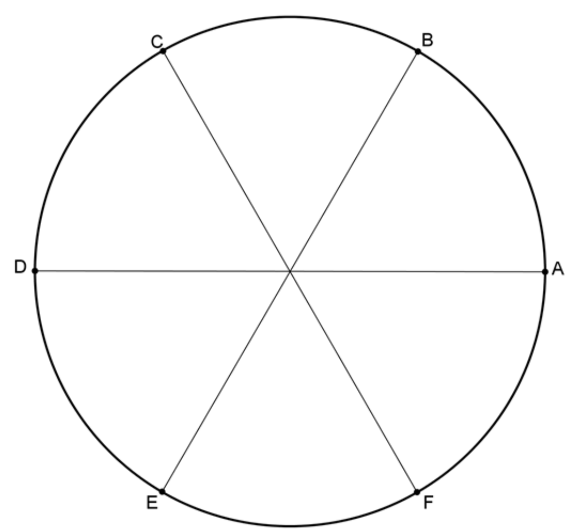


Label the figures below using a similar approach as in question 13.

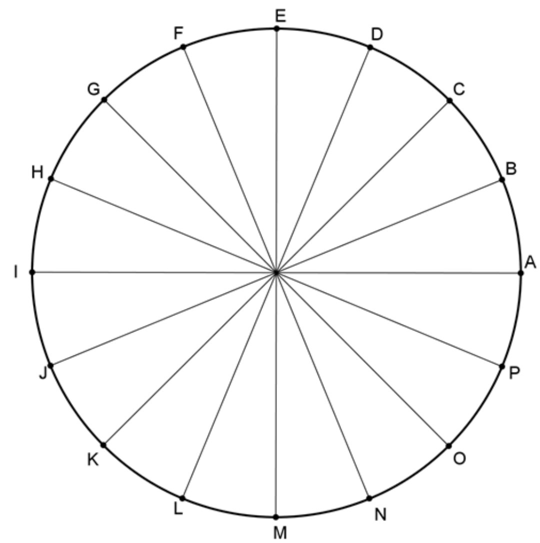
14.



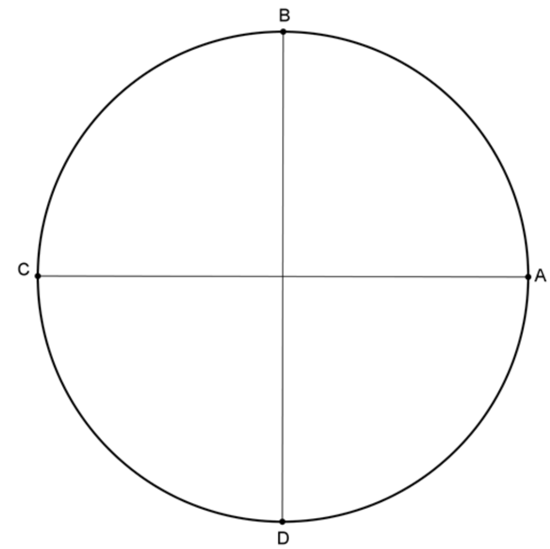
15.



16.



17.



Name: **Trigonometric Functions | 6.3**

Ready, Set, Go!

Ready
Topic: Reducing complex fractions

Write each of the following as a simple fraction. Rationalize the denominators when appropriate.

1. $\dfrac{\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}}$

2. $\dfrac{\frac{8\sqrt{3}}{5}}{\frac{1}{5}}$

3. $\dfrac{8}{\frac{1}{2}}$

4. $\dfrac{\frac{7\sqrt{3}}{2}}{\frac{1}{2}}$

5. $\dfrac{1}{\sqrt{2}}$

6. $\dfrac{3}{\sqrt{3}}$

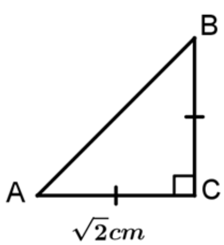
7. $\dfrac{4}{\sqrt{8}}$

8. $\dfrac{\frac{2}{3}}{\frac{1}{2}}$

9. $\dfrac{\frac{2}{\sqrt{7}}}{\frac{5}{\sqrt{7}}}$

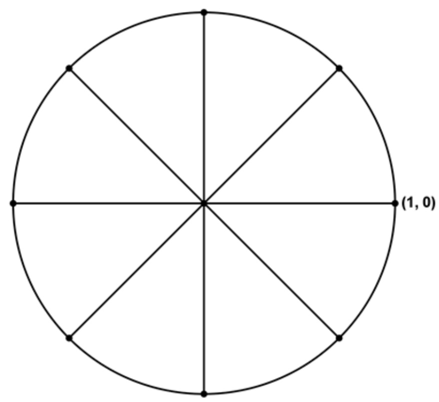
Set
Topic: Radian measure of an angle

10. ΔABC is an isosceles right triangle. The length of one side is given. Fill in the values for the missing sides and angles A and B.

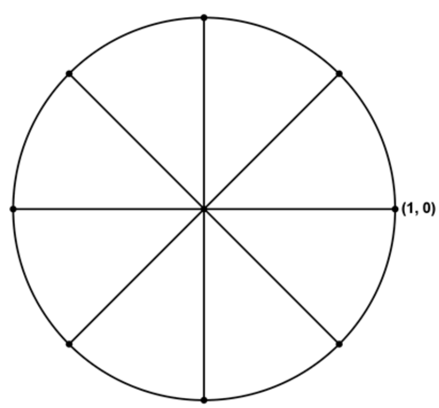


11. Use ΔABC above to find the values of $\sin A$ and $\cos A$ (leave your answers as a fraction).

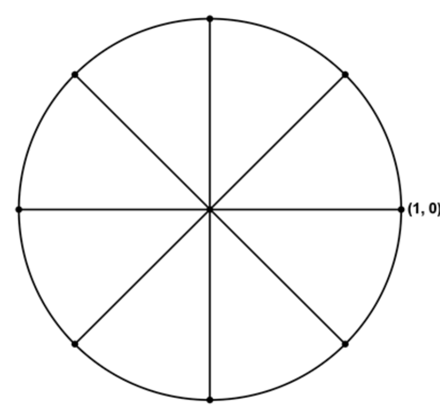
12. Label each point around the circle with the angle of rotation in radians starting from the point $(1, 0)$.



13. Use the values in #12 to write the **exact** coordinates of the points on the circle below. Be mindful of the numbers that are negative.



14. Find the arc length, *s,* from the point $(1, 0)$ to each point around the circle. Record your answers as decimal approximations to the nearest thousandth.



Use your calculator to find the following values. Round to the nearest thousandth.

15. $\sin\frac{5\pi}{4} =$

16. $\sin\frac{7\pi}{4} =$

17. Why are both of your answers in questions 15 & 16 negative?

18. $\cos\frac{\pi}{4} =$

19. $\cos\frac{7\pi}{4} =$

20. Why are both of your answers in questions 18 & 19 positive?

21. $\sin\frac{3\pi}{4} =$

22. $\cos\frac{3\pi}{4} =$

23. Why is one answer positive and one answer negative?

Go

Topic: Trigonometric values in the special triangles

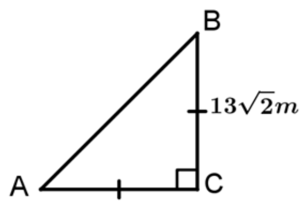
Angle C is the right angle in each of the triangles below. **Use the given information to find the missing sides and the missing angles. Then find the indicted trigonometric values.** Rationalize denominators when appropriate. Do NOT change the values to decimals. Square roots are **exact** values. Decimal representations of the square roots are approximations.

24.

$\sin A =$

$\cos A =$

$\tan A =$

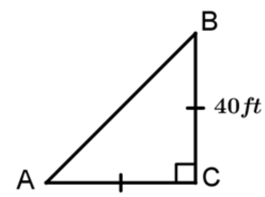


25.

$\sin B =$

$\cos B =$

$\tan B =$



26. Explain why the trigonometric values were the same for angle A and angle B even though the dimensions of the triangles were different.

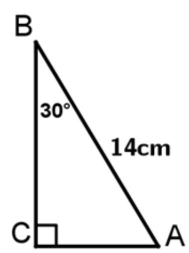
Find the values of each trigonometric function using the provided special right triangles.

27.

$\sin B =$

$\cos B =$

$\tan B =$

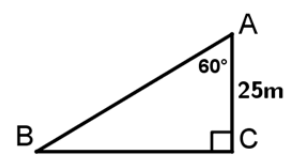


28.

$\sin A =$

$\cos A =$

$\tan A =$

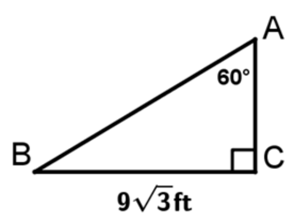


29.

$\sin A =$

$\cos A =$

$\tan A =$

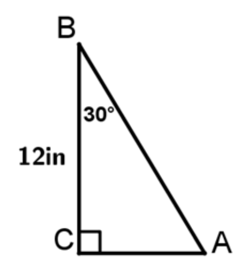


30.

$\sin B =$

$\cos B =$

$\tan B =$



31. Explain where you see the meaning of the identity $\sin\theta = \cos(90° - \theta)$ in problems 27, 28, 29, and 30.

Name: **Trigonometric Functions | 6.4**

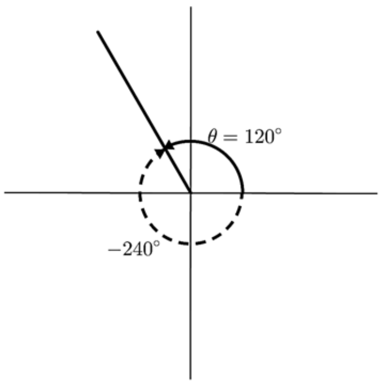
Ready, Set, Go!

Ready

Topic: Coterminal angles

State a negative angle of rotation that is *coterminal* with the given angle of rotation. ***Coterminal*** angles share the same terminal side of an angle of rotation. Sketch and label both angles.

Example: $\theta = 120°$ is the given angle of rotation. The angle of rotation is indicated by the solid arc. The dotted angle of rotation is the coterminal angle with a rotation of $-240°$.



1. Given $\theta = 20°$
   Coterminal Angle ___________________

2. Given $\theta = 95°$
   Coterminal Angle ___________________

3. Given $\theta = 225°$
   Coterminal Angle ___________________

4. Given $\theta = 270°$
   Coterminal Angle ________________

5. Given $\theta = 300°$
   Coterminal Angle ________________

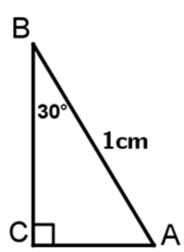
6. What is the sum of a positive angle of rotation and the absolute value of its negative coterminal angle?

7. Every angle has an infinite number of coterminal angles both positive and negative if the definition is extended to angles of rotation greater than 360°. For example: an angle of 45° is coterminal with angles of rotation measuring 405°, 765° etc. Given $\theta = 115°$, name 3 **positive** coterminal angles.

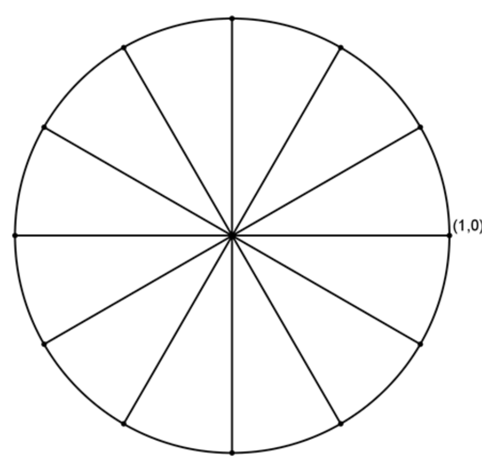
Set

Topic: Sine and cosine of radian measures

8. ΔABC is a $30° - 60° - 90°$ right triangle. The length of one side is given. Fill in the exact values for the missing sides. $m\angle B = 30°$.

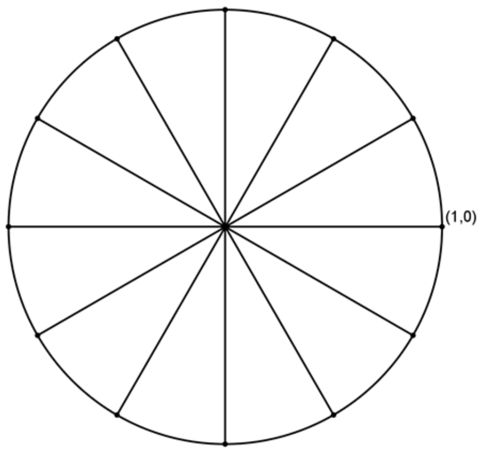


9. Label each point around the circle with the angle of rotation in radians starting from the point $(1, 0)$.

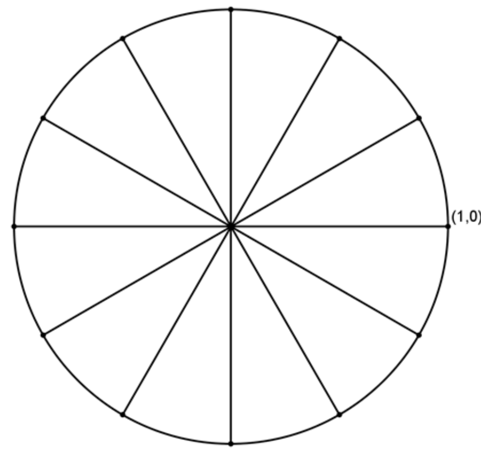


10. Use ΔABC above to find the values of $\sin A$ and $\cos A$ (leave your answers as a fraction).

11. Use the values in #9 to write the **exact** coordinates of the points on the circle below. Be mindful of the numbers that are negative.



12. Find the arc length, *s*, from the point $(1, 0)$ to each point around the circle. Record your answers as decimal approximations to the nearest thousandth.



Use your calculator to find the following values.

13. $\sin\frac{5\pi}{6} =$

14. $\sin\frac{\pi}{3} =$

15. Why are both of your answers to questions 13 & 14 positive?

16. $\cos\frac{2\pi}{3} =$

17. $\cos\frac{4\pi}{3} =$

18. Why are both of your answers to questions 16 & 17 negative?

19. $\sin\frac{\pi}{2} =$

20. $\cos\frac{\pi}{2} =$

21. In which quadrants are sine and cosine both negative?

22. Name an angle of rotation where sine is equal to -1.

23. Name an angle of rotation where cosine is equal to -1.

Go

Topic: Inverse trigonometric functions

Use your calculator to find the value of θ where $0 \leq \theta \leq 90°$. Round your answers to the nearest degree.

24. $\sin\theta = 0.82$

25. $\cos\theta = 0.31$

26. $\cos\theta = 0.98$

27. $\sin\theta = 0.39$

28. $\sin\theta = 1$

29. $\cos\theta = 0.71$

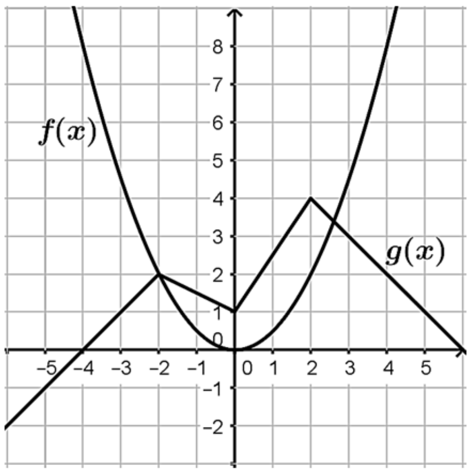
Name: **Trigonometric Functions | 6.5**

Ready, Set, Go!

Ready

Topic: Using graphs to evaluate function expressions

Use the graph of $f(x)$ and $g(x)$ below to find the indicated values.



1. $f(-4)g(-4)$

2. $f(-2)g(-2)$

3. $2f(4)+4g(2)$

4. $g(-5)-f(-4)+g(5)+f(4)$

5. $\frac{f(2)}{g(2)}$

Set

Topic: Solving trigonometric equations

For each equation below, (a) find all solutions in the interval $[0, 2\pi)$ and (b) find all solutions without a restricted domain.

6. $2\cos^2 x-\cos x-1=0$

7. $2\sin^2 x-3\sin x+1=0$

8. $\tan^2 x = 3$

9. $2\sin x - \sqrt{2} = 0$

10. $\sqrt{3}\tan x = 0$

11. $3\sin^2 x = -4$

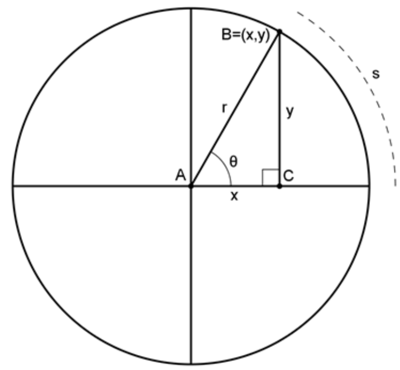
Go

Topic: Locating points in terms of rectangular coordinates, arc length, reference angle, and radius

In the diagram, ΔABC is a right triangle. Point B lies on the circle and is described by the rectangular coordinates (x, y), s is the length of the intercepted arc created by angle θ, r is the radius of circle A.

Answer the following question using the given information.

12. B is described by (r, s) where $s \approx 62.26$ and $r = 73$.



a. Find θ by using the formula $s = \frac{\theta}{360°}(d\pi)$.

b. Find (x, y) to the nearest whole number

c. Describe point B using the coordinates (r, θ).

13. A submarine at the surface of the ocean makes an emergency dive, its path making an angle of 21° with the surface.

a. If it goes 300 meters along its downward path, how deep will it be?

b. What horizontal distance is it from its starting point?

c. How many meters must it go along its downward path to reach a depth of 1000 meters?

Topic: Solving problems using right triangle trigonometry

Make a sketch of the following problems and then solve.

14. A kite is flying, attached at the end of a string that is 1500 feet long. The string makes an angle of 43° with the ground. How far above the ground is the kite? Round your answer to the nearest foot.

15. A ladder leans against a building. The top of the ladder reaches a point on the building that is 12 feet above the ground. The foot of the ladder is 4 feet from the building. Find, to the nearest degree, the measure of the angle that the ladder makes with the level ground. What is the angle the top of the ladder makes with the building?

16. The shadow of a flagpole is 40.6 meters long when the angle of elevation of the sun is 34.6°. Find the height of the flagpole. Round your answer to the nearest tenth of a meter.

Name: **Trigonometric Functions | 6.6**

Ready, Set, Go!

Ready

Topic: Revolutions and angular and linear speed

The number of degrees an object rotates during a given amount of time is called ***angular speed***. For instance, the minute hand on a clock has an angular speed of $\frac{360°}{1\text{ hr}}$ while the hour hand on a clock only rotates 30° during the hour. Therefore, the hour hand has an angular speed of $\frac{30°}{1\text{ hr}}$ or 30° per hour. Remember that a revolution is a full circle or 360°.

1. What is the angular speed of the second hand on a clock in degrees per minute? In other words, how many degrees does the second hand rotate in one minute?

2. What is the angular speed of the minute hand on a clock in degrees per minute? degrees per second?

3. What is the angular speed of the hour hand in degrees per second?

Your grandparents probably enjoyed music just as much as you do, but they didn't have i-Pods or MP3 players. They had vinyl records and phonographs. Vinyl records came in 3 speeds. A record could be a 45, $33\frac{1}{3}$, or a 78. These numbers referred to the **rpms** or ***revolutions per minute***.

4. Calculate the *angular speed* of a 45 rpm, $33\frac{1}{3}$ rpm, and 78 rpm record in degree per minute.

   a. 45 rpm

   b. $33\frac{1}{3}$ rpm

   c. 78 rpm

Angular speed describes how fast something is turning. ***Linear speed*** describes how far it travels while it is turning. ***Linear speed*** **depends on the circumference of a circle ($C = 2\pi r$) and the number of revolutions per minute.**

Vinyl records were not the same size. A 45 rpm record had a diameter of 7 inches, a $33\frac{1}{3}$ rpm record had a diameter of 12 in. and a 78 rpm record had a diameter of 10 inches.

5. a. If a fly were sitting on the outer edge of a 45 rpm record, how far would it travel in one minute?

   b. How far for a $33\frac{1}{3}$ rpm record?

   c. How far for a 78 rpm record?

Set

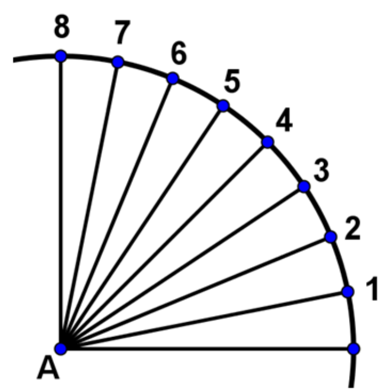
Topic: Using trigonometric ratios to solve problems

Perhaps you have seen *The London Eye* in the background of a recent James Bond movie or on a television show. When it opened in March of 2000, it was the tallest Ferris wheel in the world. The passenger capsule at the very top is 135 meters above the ground. The diameter is 120 meters.

6. How high off the ground is the center of the Ferris wheel?

7. How far from the ground is the very bottom passenger capsule?

8. Assume there are 32 passenger capsules, evenly spaced around the circumference. Find the height from the ground of each of the ***even numbered*** passenger capsules shown in the figure. Use the figure at the right to help you think about the problem.



Go

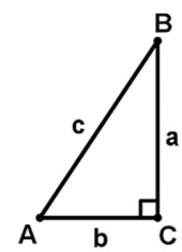
Topic: Trigonometric ratios in a right triangle

Find the other two trigonometric ratios based on the one that is given. Hint: draw and label a right triangle using the given trigonometric ratio.

9. $\sin\theta = \frac{4}{5}$	$\cos\theta =$	$\tan\theta =$
10. $\sin\theta =$	$\cos\theta = \frac{5}{13}$	$\tan\theta =$
11. $\csc\theta =$	$\sec\theta =$	$\cot\theta = 1$
12. $\sin\theta = \frac{1}{2}$	$\sec\theta =$	$\cot\theta =$
13. $\csc\theta =$	$\cos\theta = \frac{9}{41}$	$\sec\theta =$
14. $\sin\theta =$	$\cos\theta =$	$\tan\theta = \sqrt{3}$

15. Find all six trigonometric ratios of angle A in ΔABC. Leave answers in terms of a, b, and/or c.

.



Name: **Trigonometric Functions 6.7**

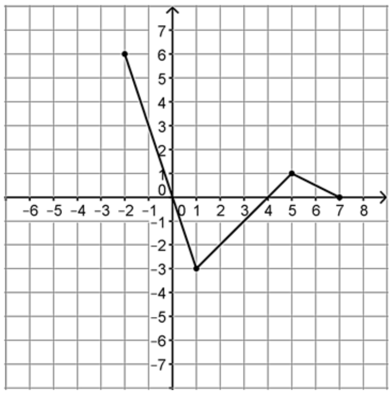
Ready, Set, Go!

Ready

Topic: Describing intervals of graphs

Write the intervals where the graph is positive and the intervals where the graph is negative.

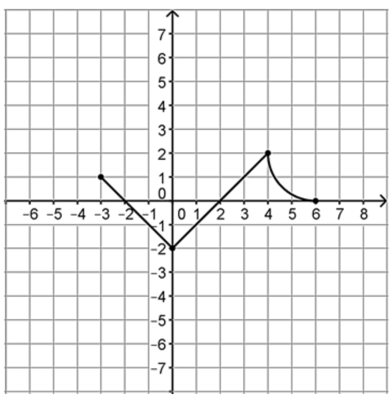
1.



Positive: ______________________________

Negative: ______________________________

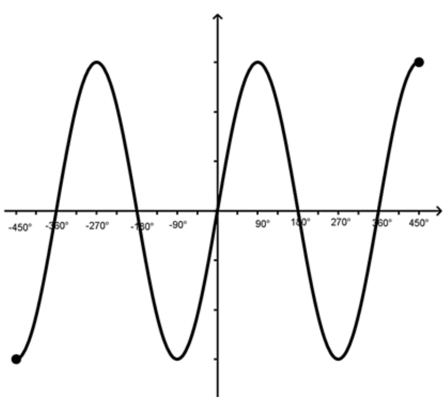
2.



Positive: ______________________________

Negative: ______________________________

3.

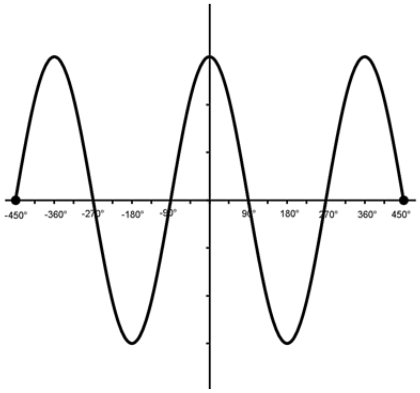


(The scale on the *x*-axis is in 45° increments.)

Positive: ______________________________

Negative: ______________________________

4.



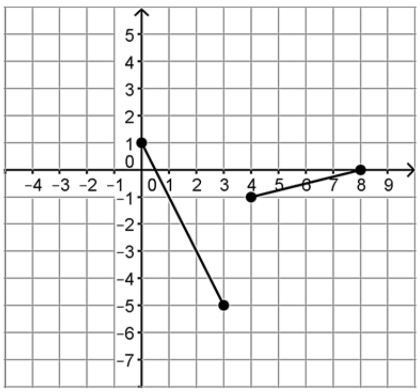
(The scale on the x-axis is in 45° increments.)

Positive: ________________________________

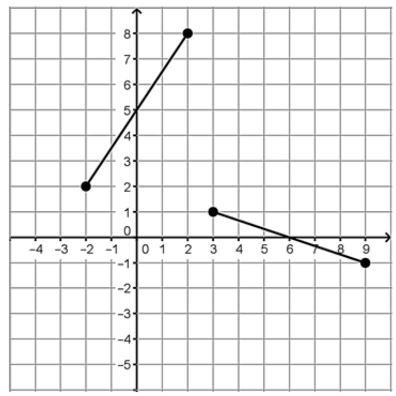
Negative: ________________________________

Write the piecewise equations for the given graphs.

5.

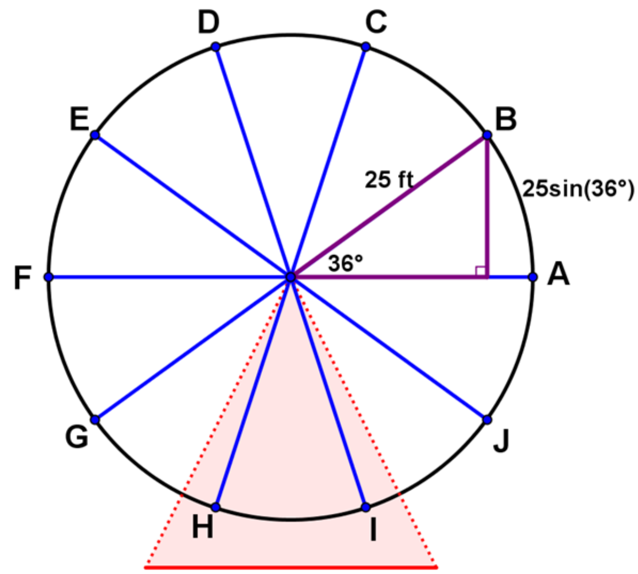


6.



Set

Topic: Sine as a function of time



Recall the following facts from the classroom task:

- The Ferris wheel has a radius of 25 feet.
- The center of the Ferris wheel is 30 feet above the ground.

Due to a safety concern, the management of the amusement park decides to slow the rotation of the Ferris wheel from 20 seconds for a full rotation to **30 seconds for a full rotation**.

7. Calculate how high a rider will now be 2 seconds after passing position A on the diagram.

8. Complete the table below, where time represents the number of seconds since the rider passed position A on the diagram. As you calculate each height, plot the position on the diagram. Connect the center of the circle to the point you plotted. Then draw a vertical line from the plotted point on the Ferris wheel to the line segment AF in the diagram. Each time you should get a right triangle similar to the one in the figure.

Elapsed time since passing position A	**Distance of the rider from midline ($\overline{AF}$)**	**Height of the rider**
1 sec		
3 sec		
5 sec		
7 sec		

9. Find the time it takes for the rider to reach the top of the Ferris wheel. How high off the ground is the rider?

10. Using the values for time and distance the rider is from the midline ($\overline{AF}$) you found in questions 7 through 9, along with symmetry, to find the corresponding times and heights for the rider in the remaining quadrants.

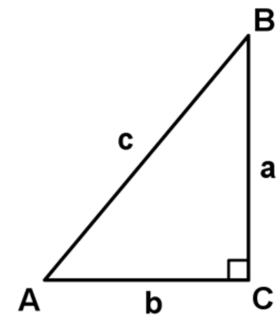
Elapsed time since passing position A	**Distance of the rider from midline ($\overline{AF}$)**	**Height of the rider**
8 sec		
10 sec		
12 sec		
13 sec		
14 sec		
16 sec		
17 sec		
18 sec		
20 sec		
22 sec		
23 sec		
25 sec		
27 sec		
28 sec		
29 sec		

11. What do you notice about the height of the rider as time passes?

12. What do you expect to happen to the height as time continues past 29 seconds? Why?

Go

Topic: Finding missing angles in triangles



Find the measure of each acute angle for the triangle at the right. Round your answers to the nearest degree.

13. $a = 3$ inches $c = 5$ inches

14. $a = 5$ feet $c = 10$ feet

15. $a = 9.1$ cm $c = 12.3$ cm

16. $a = 14.1$ cm $c = 18$ cm

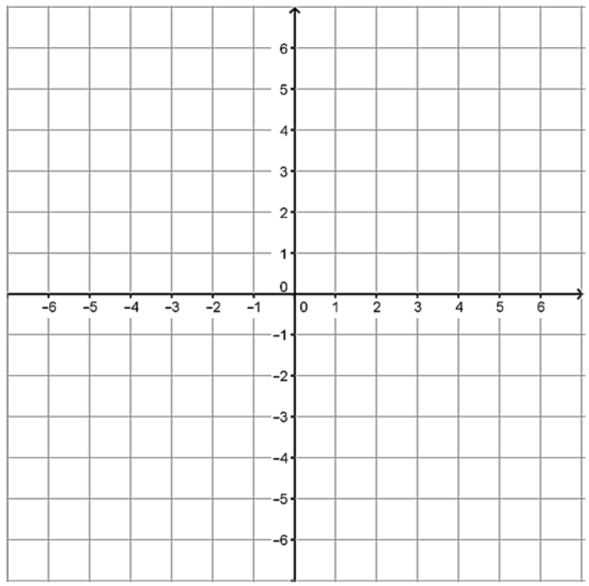
Ready, Set, Go!

Ready

Topic: Graphing a curve

1. Graph the table of values. Connect your points with a smooth curve.

x	y
-6	0
-5	-3
-4	-4
-3	-3
-2	0
-1	3
0	4
1	3
2	0
3	-3
4	-4
5	-3
6	0



2. Identify the relative/local maximum and minimum values of the curve.

3. This curve repeats itself. (It's called a ***periodic function***.) Find the length of the interval that would allow you to see **exactly** one full curve without repetition.

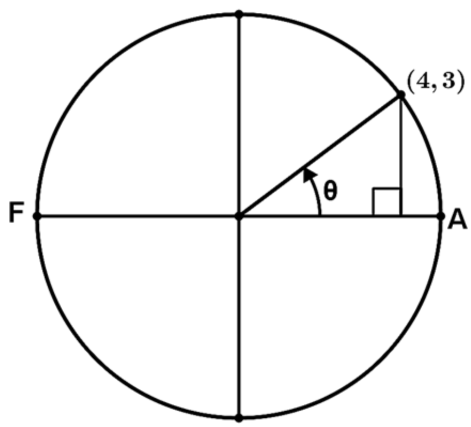
4. The curve is positive on the interval $(-2, 2)$. Identify two more intervals where this curve will be positive.

Set

Topic: Values of sine in the coordinate plane

Use the given point on the circle to find the value of sine. Recall that $r = \sqrt{x^2 + y^2}$ and $\sin\theta = \frac{y}{r}$.

5.



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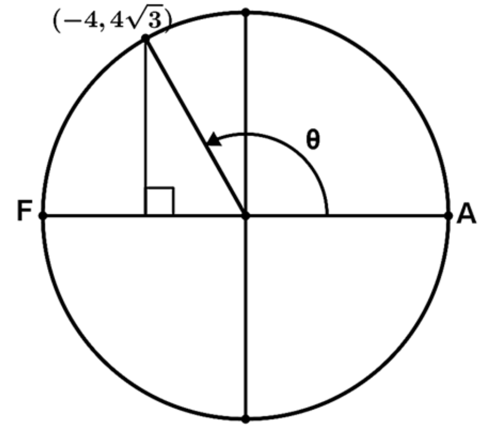

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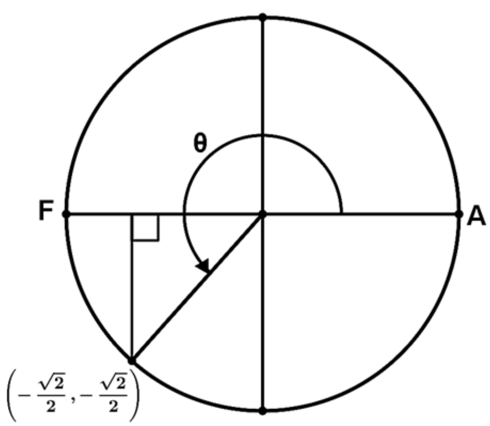

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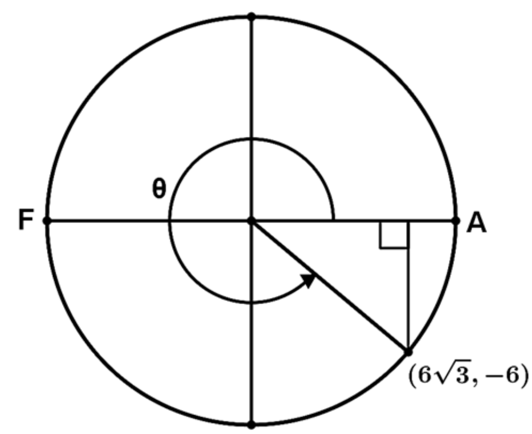
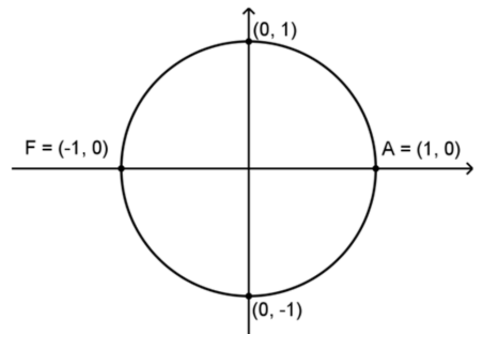


9. In each graph above, the angle of rotation is indicated by an arc and θ. Describe the angles of rotation that make the y-values of the points be positive and the angles of rotation that make the y-values be negative.

10. In the graph at the right, the radius of the circle is 1. The intersections of the circle and the axes are labeled. Based on your observations from the task, what is the value of sine for:
    90°? 180°? 270°? 360°?



Go

Topic: Solving problems using right triangle trigonometry

Make a sketch of the following problems and then solve.

11. A kite is flying, attached at the end of a string that is 1500 feet long. The string makes an angle of 43° with the ground. How far above the ground is the kite? Round your answer to the nearest foot.

12. A ladder leans against a building. The top of the ladder reaches a point on the building that is 12 feet above the ground. The foot of the ladder is 4 feet from the building. Find, to the nearest degree, the measure of the angle that the ladder makes with the level ground. What is the angle the ladder makes with the building?

13. The shadow of a flagpole is 40.6 meters long when the angle of elevation of the sun is 34.6°. Find the height of the flagpole. Round your answer to the nearest tenth of a meter.

14. The angle of depression from the top of a building to a car parked in the parking lot is 32.5°. How far from the top of the building is the car on the ground, if the car is 330 meters from the bottom of the building? Round your answer to the nearest tenth of a meter.

Name **Trigonometric Functions | 6.9**

Ready, Set, Go!

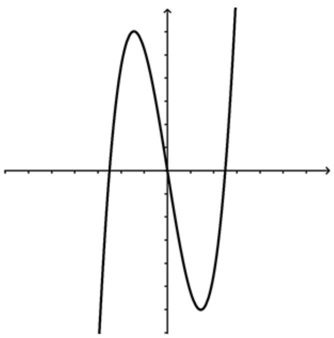
Ready

Topic: Even and odd functions

The graphs of even and odd functions make it easy to identify the type of function. **Even** functions have a line of symmetry along the *y*-axis, while **odd** functions have 180° rotational symmetry (symmetric about the origin).

Label the following functions as even, odd, or neither.

1.



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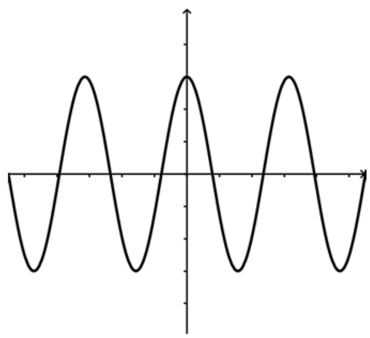

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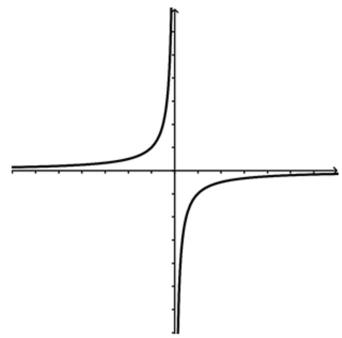

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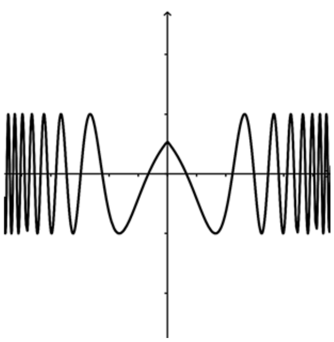

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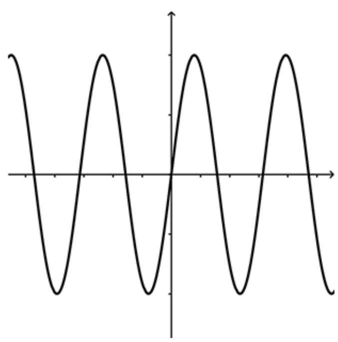

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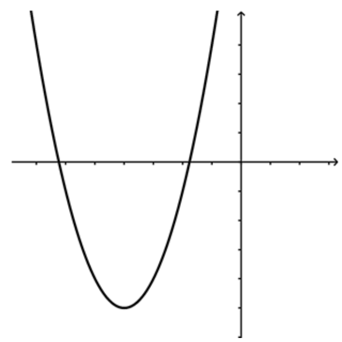


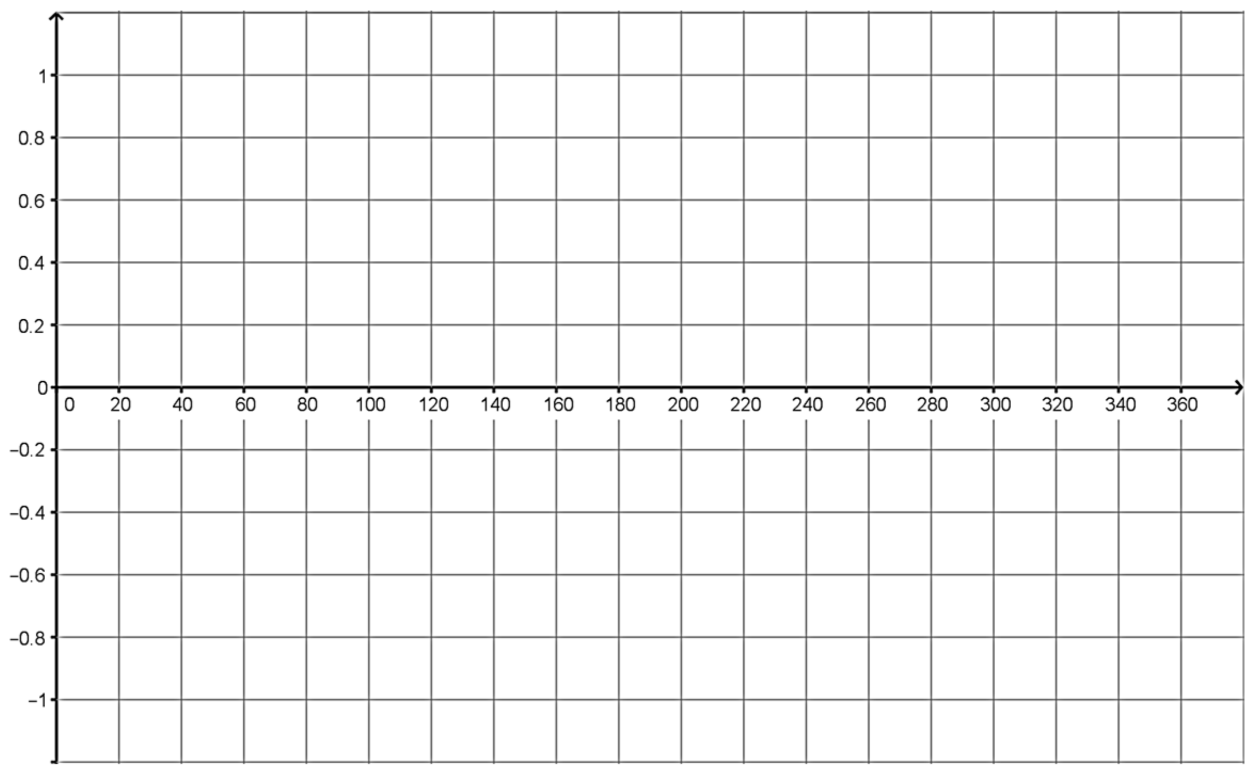
Set

Topic: Transformations on functions

Describe the transformation(s) on the parent graph in the following equations.

10. $y = x^2 + 5$	11. $y = x^2 - 1$	12. $y = -x^2$
13. $y = 4x^2$	14. $y = -3 + \log(x + 5)$	15. $y = -2\|x - 7\| + 2$
16. $y = 2^{x+6} - 4$	17. $y = \frac{1}{2}\|x + 3\| + 1$	

18. Sketch the graph of a "Ferris wheel" that has a radius of 1 unit, with center at a height of 0, and makes one complete rotation in 360 seconds.

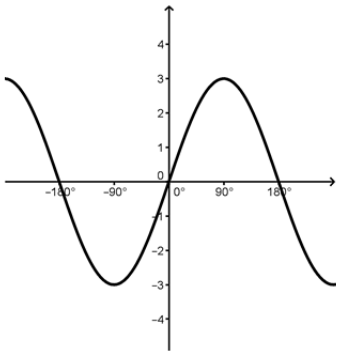


Match the equation with the correct graph.

A. $y = \sin 2x$ B. $y = (\sin x) + 2$ C. $y = 3\sin x$

D. $y = -(\sin x\,) - 2$ E. $y = -2\sin x$ F. $y = 3\sin 2x$

19.



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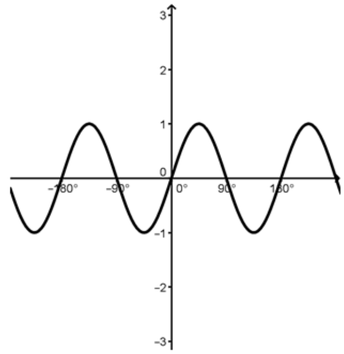

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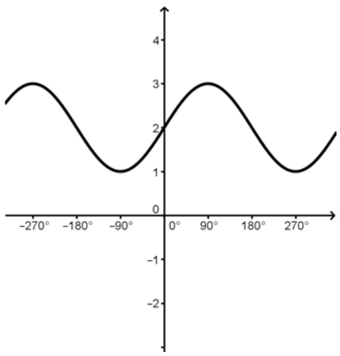

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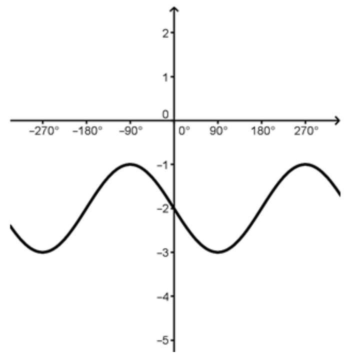

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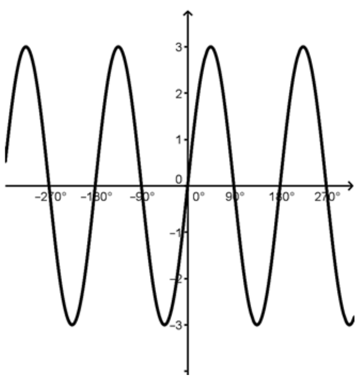

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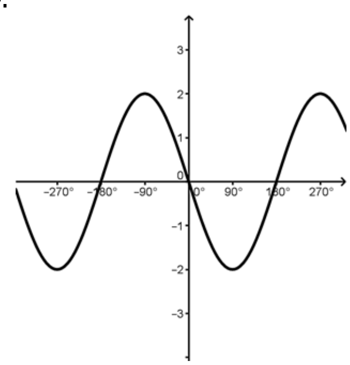
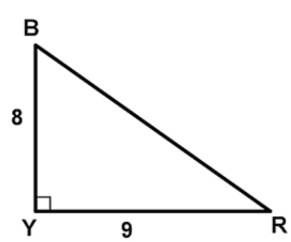


Go

Topic: Trigonometric ratios

Use the triangle at the right to find each trigonometric ratio.

25. $\sec B =$ 26. $\cot R =$ 27. $\csc B =$



Ready, Set, Go!

Ready

Topic: Exact values of cosine

Find the exact values of each.

1. $\cos\frac{2\pi}{3} =$

2. $\cos -\frac{\pi}{6} =$

3. $\cos\frac{\pi}{4} =$

4. $\cos -\frac{7\pi}{3} =$

5. $\cos\frac{\pi}{6} =$

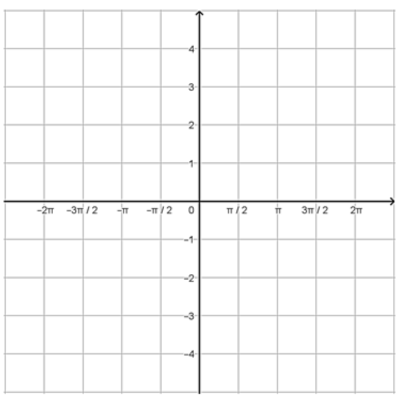
6. $\cos\frac{3\pi}{4} =$

Set

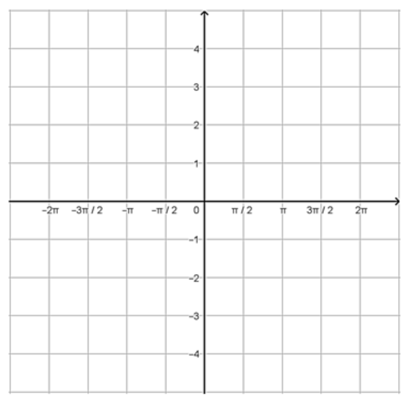
Topic: Graphing transformations of $y = \sin x$

For each function below, state the various transformations of $y = \sin x$ and graph two complete periods.

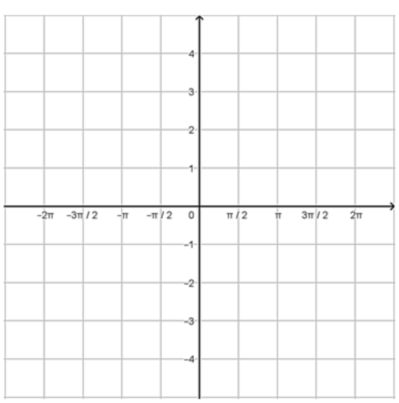
7. $f(x) = 4\sin x$



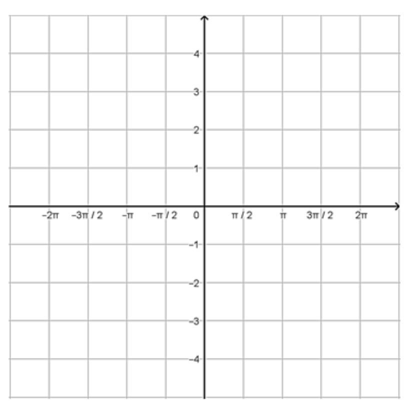
8. $f(x) = 3\sin x + 2$



9. $f(x) = 2\sin(x - \pi)$

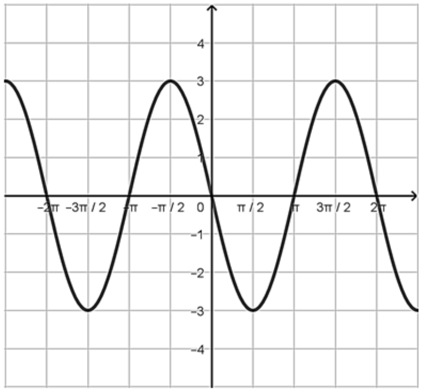


10. $f(x) = \sin(2x)$



For each graph given, state a possible equation involving transformation(s) of $y = \sin x$.

11.



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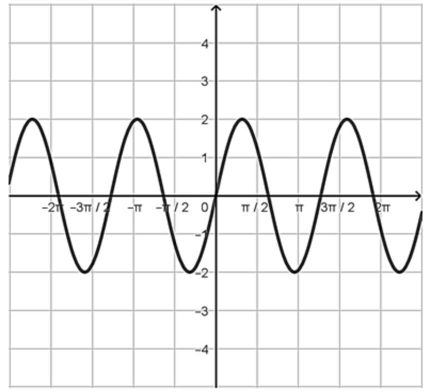


Go

Topic: Solving trigonometric equations

Solve each equation on the interval $[\mathbf{0}, \mathbf{2\pi})$

13. $\cos x = -1$

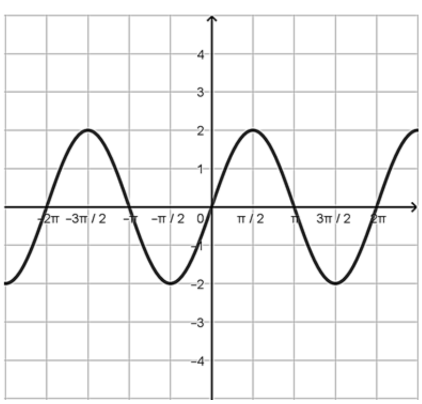
14. $\tan x - 2\cos x \tan x = 0$

15. $\cos^2 x - 2\cos x - 3 = 0$

Topic: Verifying trigonometric identities

16. Half of a trigonometric identity and its graph are given. Make a conjecture as to what the right side of the identity should be. Then prove your conjecture.

$\frac{\cos x + \cot x \sin x}{\cot x} = ?$



Ready, Set, Go!

Ready

Topic: Comparing radius and arc length

The wheels on the wagons that the pioneers used to cross the plains were smaller in the front than in the back. The front wheel had 12 spokes. The top of the front wheel measured 44 inches from the ground. The rear wheel had 16 spokes. The top of the rear wheel measured 59 inches from the ground. (For these problems disregard the hub at the center of the wheel. Assume the spokes meet in the center at a point.)

Find the following values. Express answers in terms of π.

1. Find the area and the circumference of each wheel.

2. Determine the central angle between the spokes on each wheel.

3. Find the distance on the circumference between two consecutive spokes for each wheel.

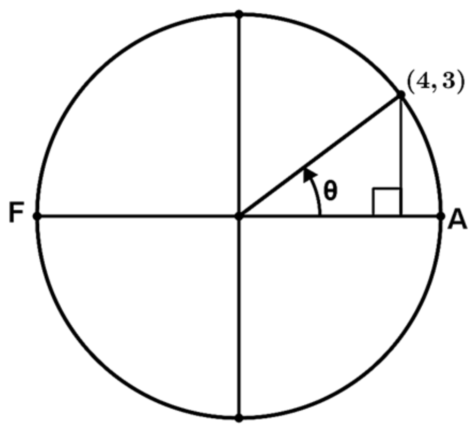
4. A wheel rotates r times per minute. Describe how to find the number of degrees the wheel rotates in t seconds.

Set

Topic: Values of cosine in the coordinate plane

Use the given point on the circle to find the value of cosine. Recall that $r = \sqrt{x^2 + y^2}$ and $\cos\theta = \frac{x}{r}$.

5.



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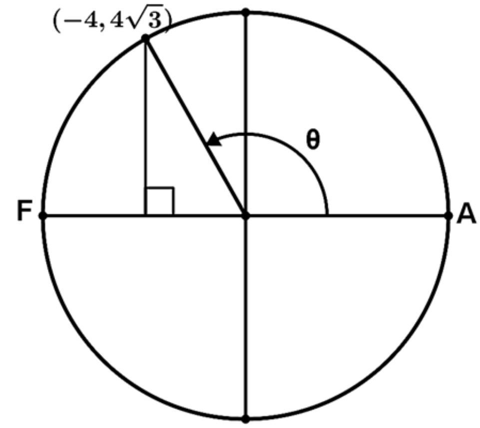

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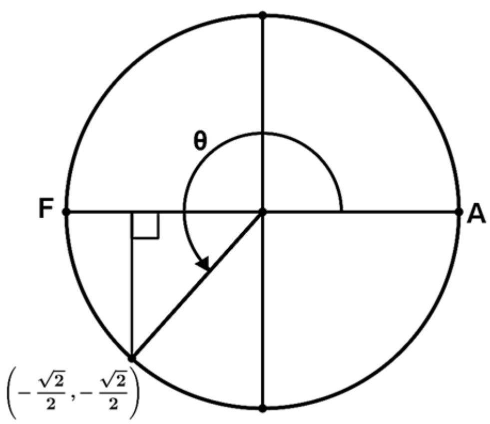

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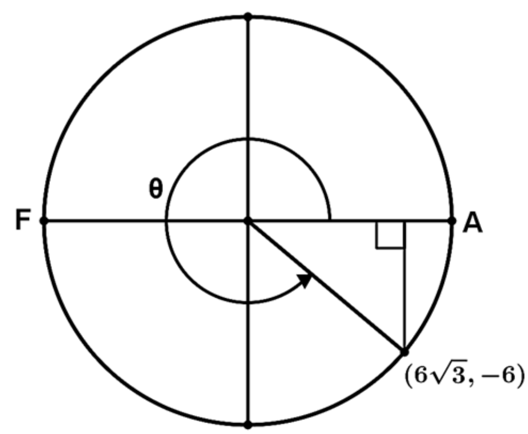
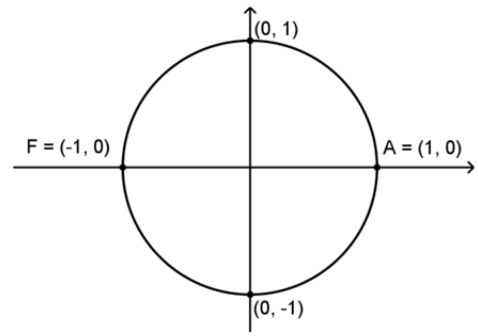


9. In each graph, the angle of rotation is indicated by an arc and θ. Describe the angles of rotation that make the x-values of the points be positive and the angles of rotation that make the x-values be negative.

10. In the graph at the right, the radius of the circle is 1. The intersections of the circle and the axes are labeled. Based on your observation in #9, what is the value of cosine for:
    90°? 180°? 270°? 360°?

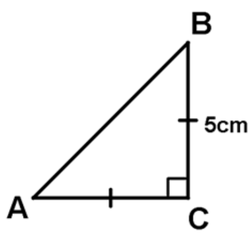


Go

Topic: Measures in special triangles

ΔABC is a right triangle. Angle C is the right angle. Use the given information to find the missing sides and the missing angles.

11.



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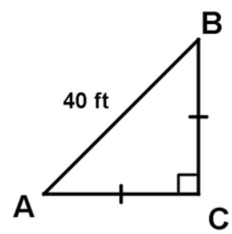

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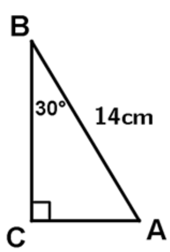

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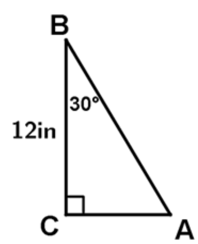
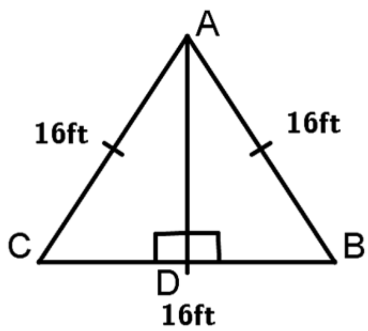


Find AD in the figures below:

15.



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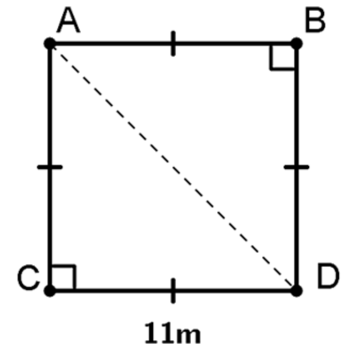
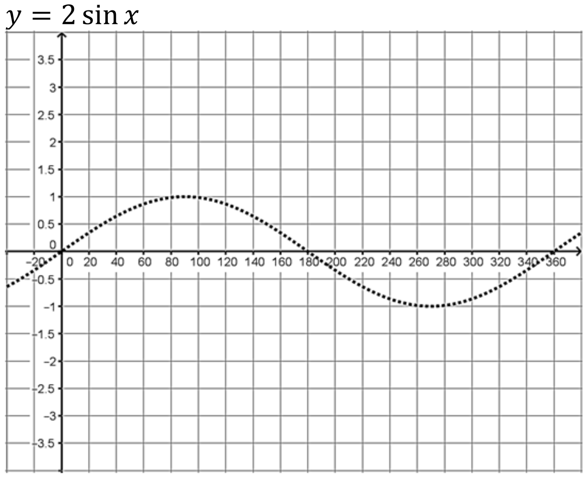


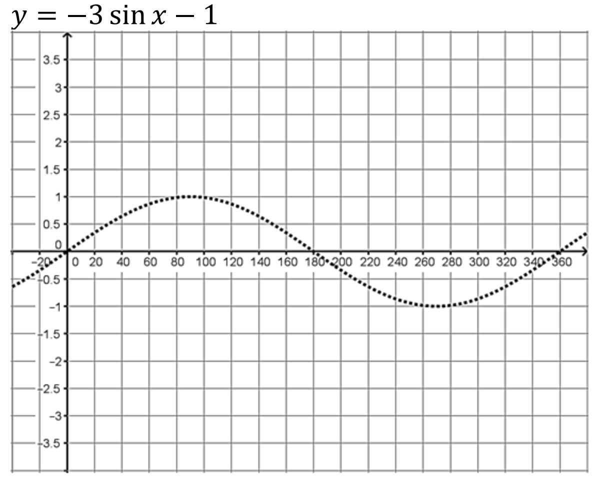
Topic: Graphing trigonometric functions

Graph the following functions (the graph of $y = \sin x$ is given to assist you).

17. $y = 2\sin x$

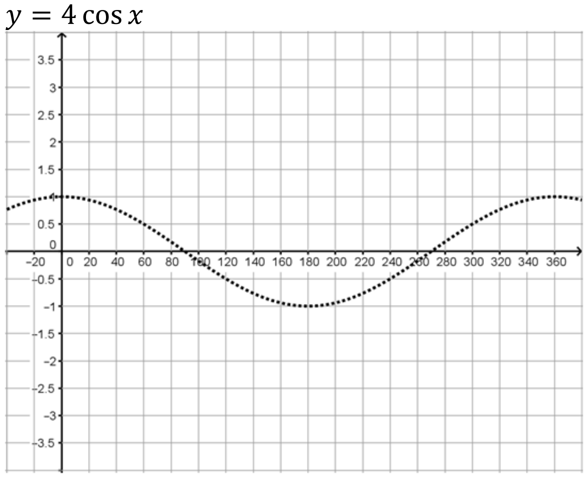


18. $y = -3\sin x - 1$

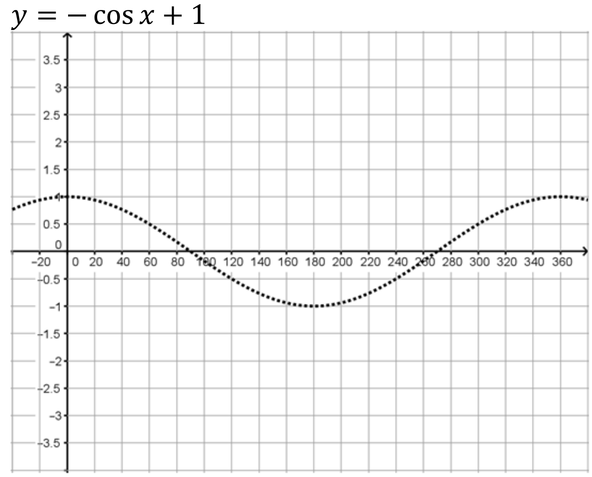


Graph the following functions (the graph of $y = \cos x$ is given to assist you).

19. $y = 4\cos x$



20. $y = -\cos x + 1$



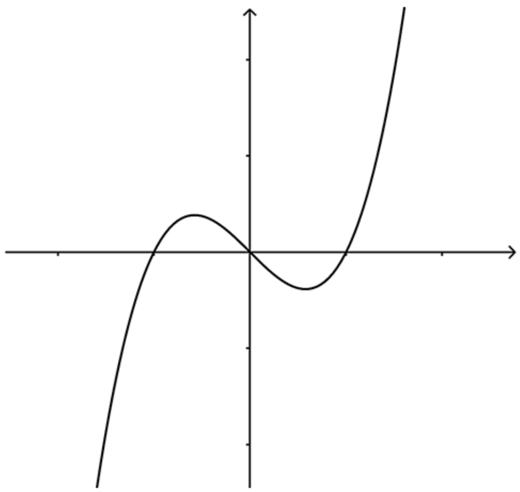
Ready, Set, Go!

Ready

Topic: Functions and their inverses

Indicate which of the following functions have an inverse that is a function. If the function has an inverse, sketch it. Remember, the inverse will reflect across the line $y = x$. Finally, label each one as *even, odd*, or *neither*. Recall that an *even* function is symmetric with respect to the *y*-axis, while an *odd* function is symmetric with respect to the origin.

1.



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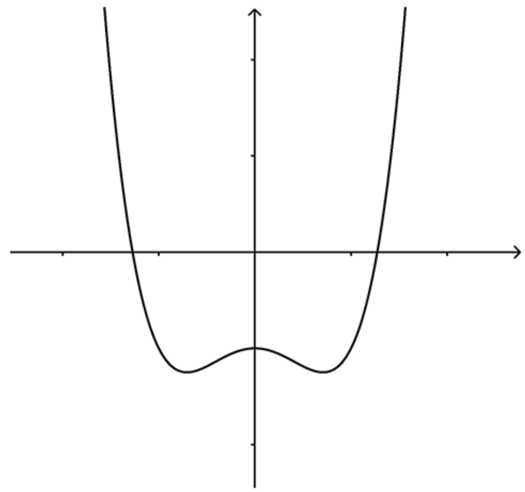

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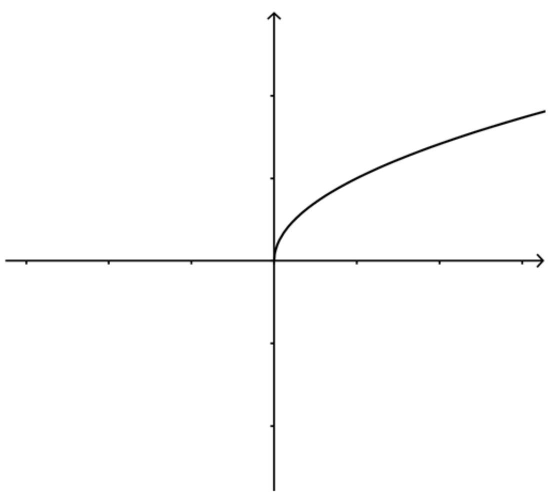

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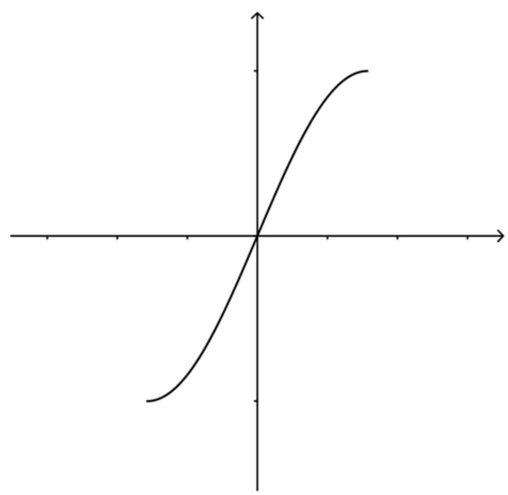
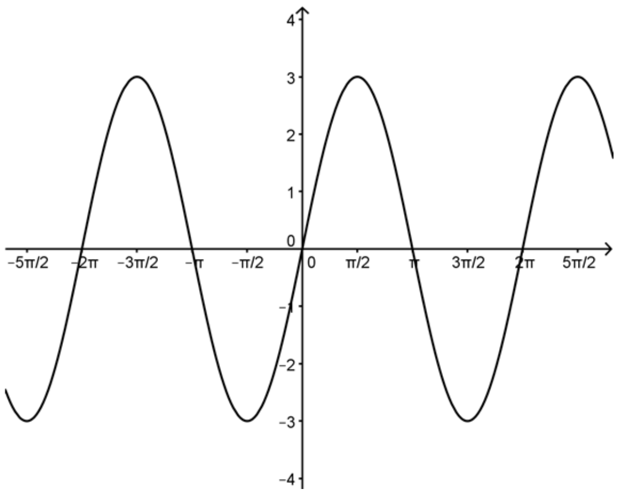


Set

Topic: Graphs of the trigonometric functions

State the period, amplitude, vertical shift, and phase shift of the function shown in the graph. Then write the equation using the given trigonometric parent function.

5. $y = \sin x$

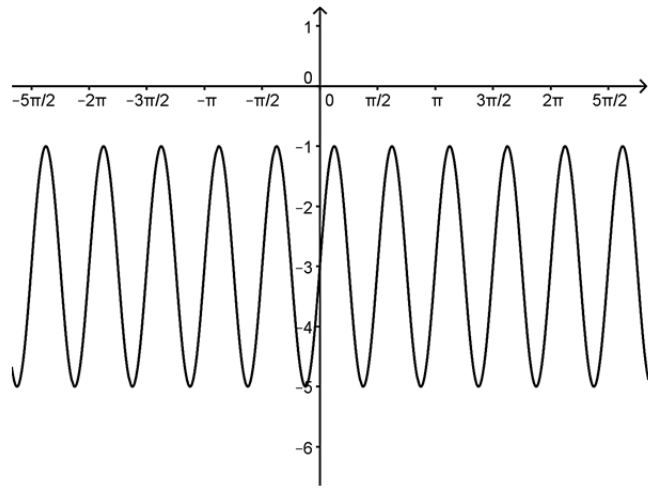


Period: Amplitude:

Vertical shift: Phase shift:

Equation:

6. $y = \sin x$

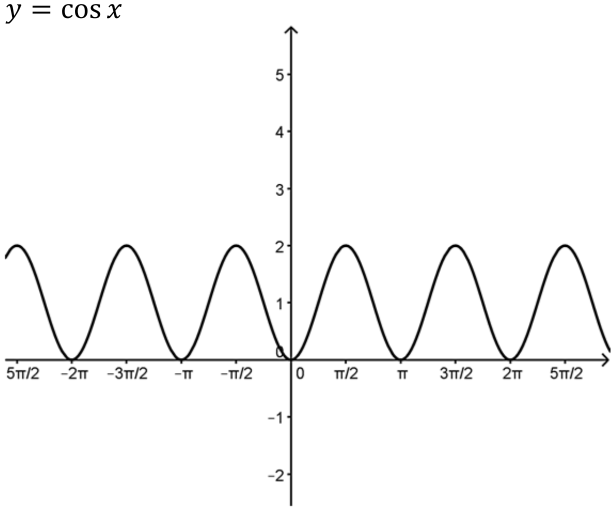


Period: Amplitude:

Vertical shift: Phase shift:

Equation:

7. $y = \cos x$

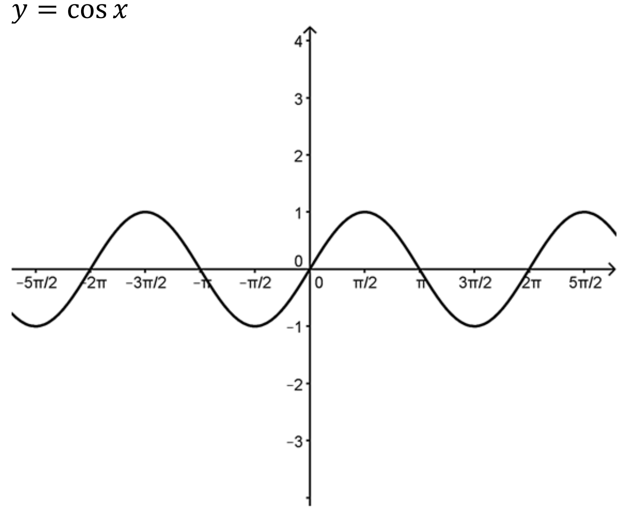


Period: Amplitude:

Vertical shift: Phase shift:

Equation:

8. $y = \cos x$

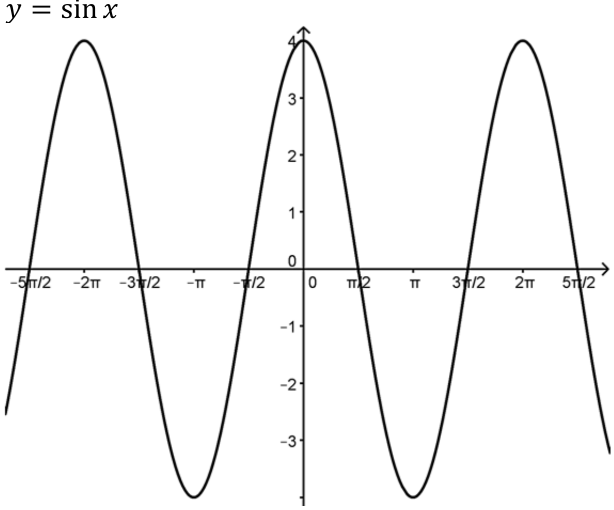


Period: Amplitude:

Vertical shift: Phase shift:

Equation:

9. $y = \sin x$



Period:

Amplitude:

Vertical shift:

Phase shift:

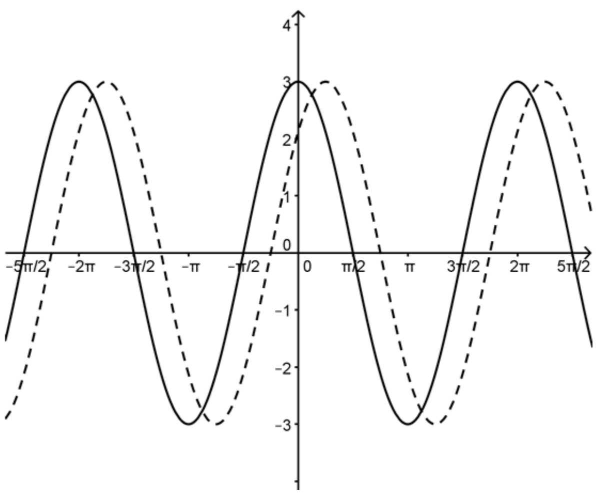
Equation:

10. The cofunction identity states that $\sin\theta = \cos(90° - \theta)$ and $\sin(90° - \theta) = \cos\theta$. How does this identity relate to the graph in #9?

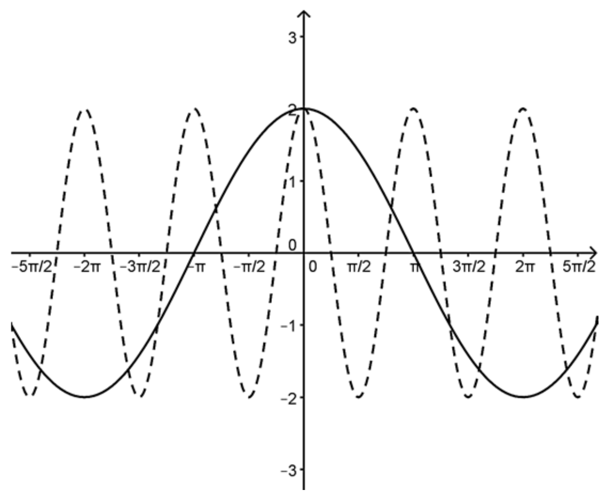
Explain where you would see this identity in a right triangle.

Describe the relationships between the graphs of $f(x)$ *(solid)* and $g(x)$ *(dotted)*. Then write their equations.

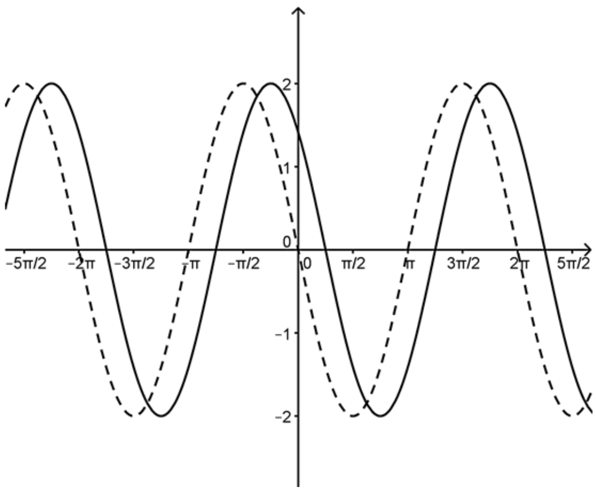
11.



12.

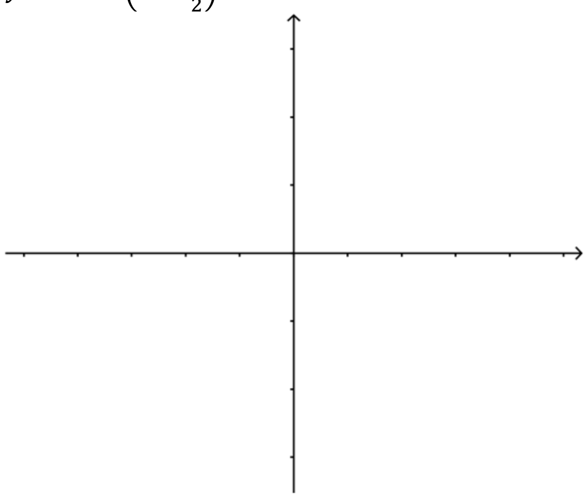


13.

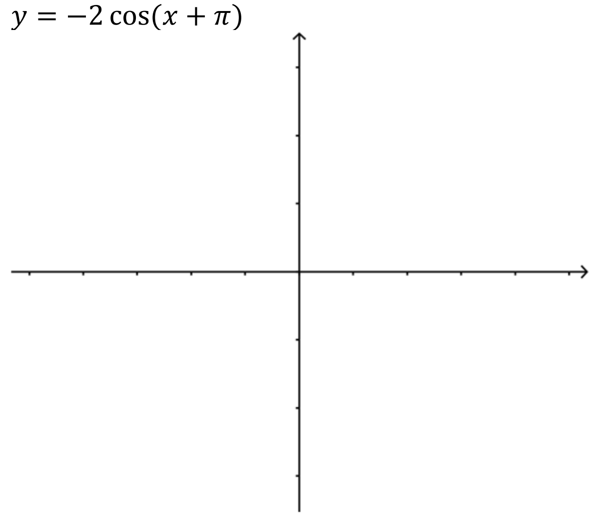


Sketch the graph of the function. Include 2 full periods. Label the scale of your horizontal axis.

14. $y = 3\sin\left(x - \frac{\pi}{2}\right)$



15. $y = -2\cos(x + \pi)$



Go

Topic: Trigonometric ratios in the unit circle

Name two values for the angle of rotation, $0 < \theta \leq 2\pi$, that have the given trigonometric ratio.

16. $\sin\theta = \frac{\sqrt{2}}{2}$

17. $\cos\theta = \frac{\sqrt{2}}{2}$

18. $\cos\theta = -\frac{1}{2}$

19. $\sin\theta = 0$

20. $\sin\theta = -\frac{\sqrt{3}}{2}$

21. $\cos\theta = -\frac{\sqrt{3}}{2}$

22. For which angles of rotation does $\sin\theta = \cos\theta$? Explain why.

Topic: Finding angle measures given trigonometric ratios in a right triangle.

Given the trigonometric values from a right triangle, find the indicated angle measures. Round angle measures to the nearest degree. Hint: draw a right triangle and label the lengths of the sides using the trigonometric ratio.

23. $\cot\theta = \frac{6}{7}$

24. $\sec\theta = 3$

25. $\csc\theta = \frac{8}{5}$

Name **Trigonometric Functions | 6.13**

Ready, Set, Go!

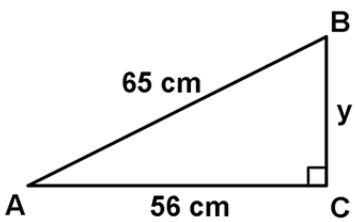
Ready

Topic: Tangents in right angle trigonometry

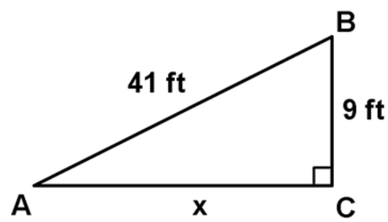
Recall that the right triangle definition of the tangent ratio is:

$$\tan A = \frac{\textbf{length of side opposite angle } A}{\textbf{length of side adjacent to angle } A}$$

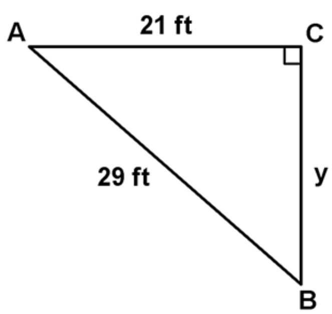
1. Find tan A and tan B



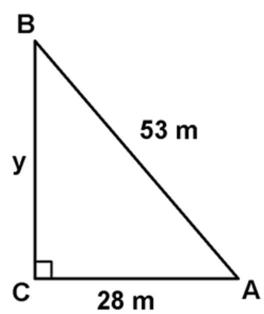
2. Find tan A and tan B



3. Find tan A and tan B



4. Find tan A and tan B



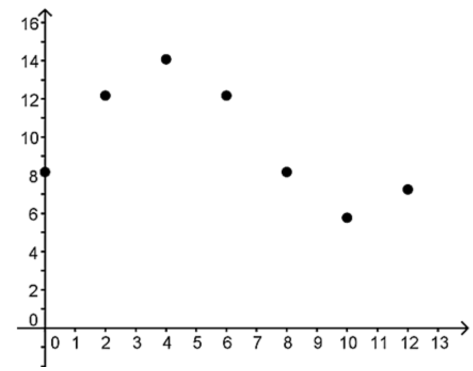
Set

Topic: Mathematical modeling using sine and cosine functions

Many real-life situations such as sound waves, weather patterns, and electrical currents can be modeled by sine and cosine functions. The table below shows the depth of water (in feet) at the end of a wharf as it varies with the tides at various times during the morning.

t (time)	midnight	2 A.M.	4 A.M.	6 A.M.	8 A.M.	10 A.M.	noon
d (depth)	8.16	12.16	14.08	12.16	8.16	5.76	7.26

We can use a trigonometric function to model the data. Suppose you choose cosine: $y = a\cos(b\,t - c) + d$, where y is the depth at any time. The ***amplitude*** will be the distance from the average of the highest and lowest values. This will be the average depth, d.



5. Sketch the line that shows the average depth, *d.*

6. Find the amplitude: $a = \frac{1}{2}(\text{high depth} - \text{low depth})$

7. Find the period: $p = 2|\text{low time} - \text{high time}|$. Since a normal period for sine is 2π, the new period for our model will be $\frac{2\pi}{p}$ so $b = \frac{2\pi}{p}$. Use the p you calculated, divide and turn it into a decimal to find the value of b.

8. High tide occurred 4 hours after midnight. The formula for the displacement is $4 = \frac{c}{b}$. Use b and solve for c.

9. Now that you have your values for *a, b, c*, and *d*, you can put them into your equation: $y = a\cos(bt - c) + d$

10. Use your model to calculate the depth at 9 A.M. and 3 P.M.

11. A boat needs at least 10 feet of water to dock at the wharf. During what interval of time in the afternoon can it safely dock?

Go

Topic: Using the calculator to find angles of rotation

Use your calculator and what you know about where sine and cosine are positive and negative in the unit circle, to find the two angles that are solutions to each equation. Make sure $\boldsymbol{\theta}$ is in the interval $\mathbf{0 < \boldsymbol{\theta} \leq 2\boldsymbol{\pi}}$. Round your answers to 4 decimal places. *(Your calculator should be set in radians.)*

You will notice that your calculator will sometimes give you a negative angle. That is because the calculator is programmed to restrict the angle of rotation so that the inverse of the function is also a function. Since the requested answers have been restricted to positive rotations, if the calculator gives you a negative angle of rotation, you will need to figure out the positive coterminal angle for the angle that your calculator has given you.

12. $\sin\theta = \frac{4}{5}$

13. $\sin\theta = -\frac{1}{10}$

14. $\sin\theta = -\frac{13}{14}$

15. $\cos\theta = \frac{11}{12}$

16. $\cos\theta = -\frac{7}{8}$

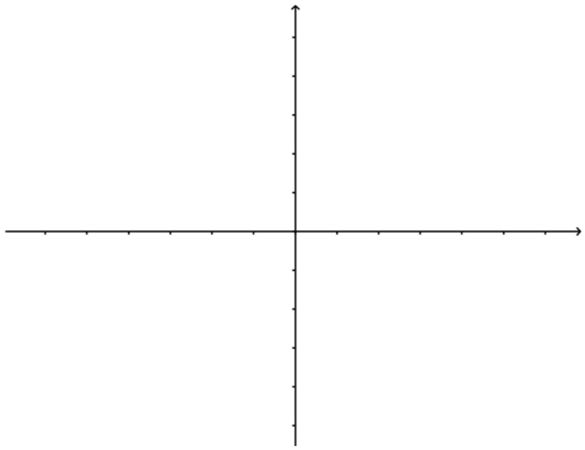
17. $\cos\theta = -\frac{2}{5}$

Note: When you ask your calculator for the angle, you are undoing the trigonometric function. Finding the angle is finding the inverse trigonometric function. When you see "$\boldsymbol{sin^{-1}\left(\frac{4}{5}\right)}$", you are being asked to find the angle that makes "$\mathbf{sin}\,\boldsymbol{\theta} = \mathbf{\frac{4}{5}}$" true. The answer would be the same as the answer your calculator gave you in #12. Another notation that represents the inverse sine function is $\mathbf{arcsin}\left(\frac{4}{5}\right)$.

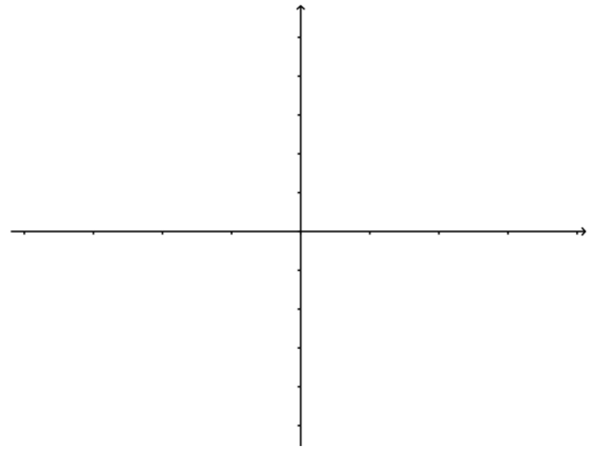
Topic: Graphing sine and cosine

Sketch the graph of each function. Include two full periods. Label the scale of your horizontal axis.

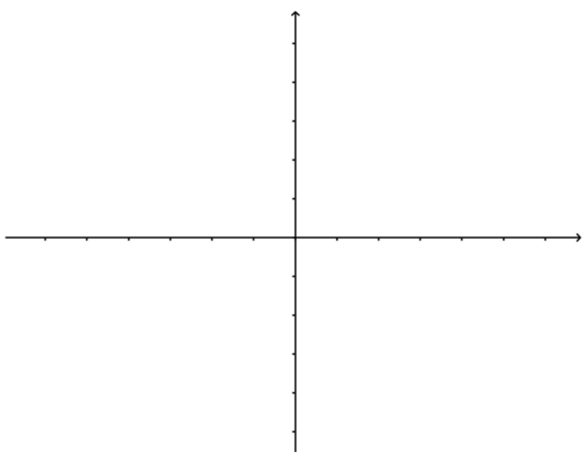
18. $y = 3\sin\left(x + \frac{\pi}{4}\right)$



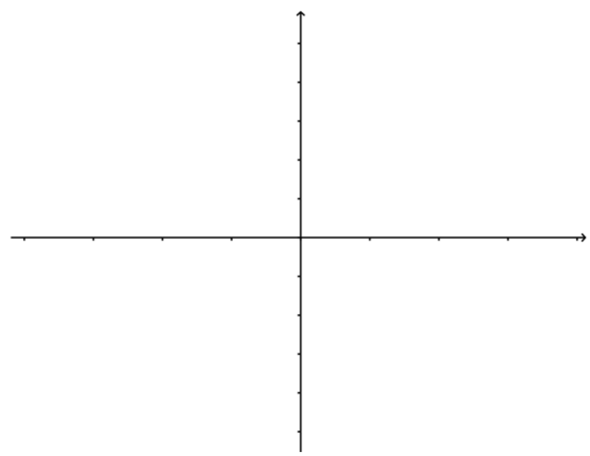
19. $y = -2\cos 3x$



20. $y = \sin 4x + 2$



21. $y = \cos\left(x - \frac{\pi}{2}\right) - 4$



Name **Trigonometric Functions 6.14**

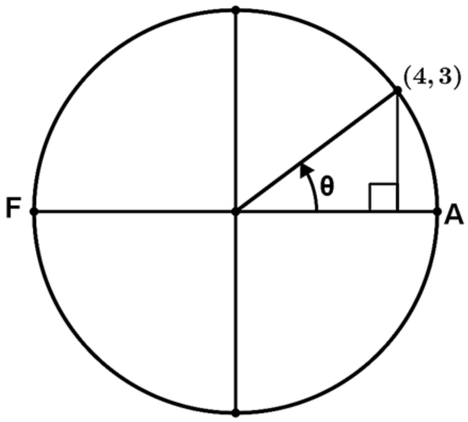
Ready, Set, Go!

Ready

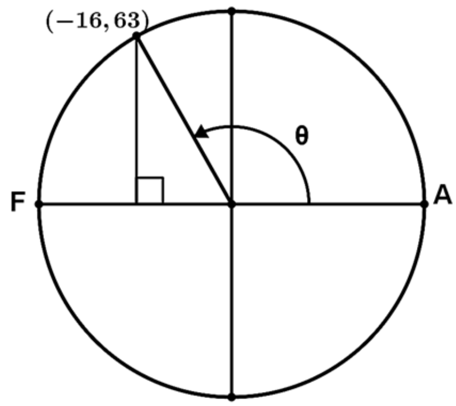
Topic: Using the definition of tangent

Use what you know about the definition of tangent in a right triangle and your work with the new definitions of sine and cosine to find the exact value of tangent θ for each of the right triangles below.

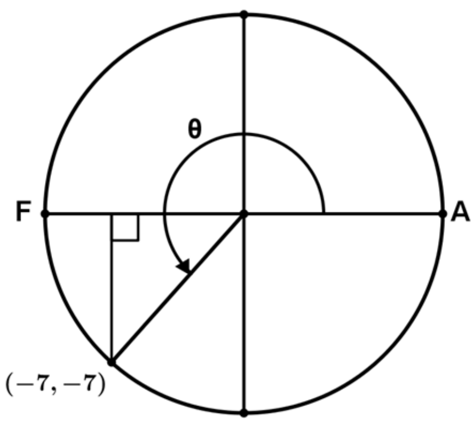
1. $\tan\theta =$



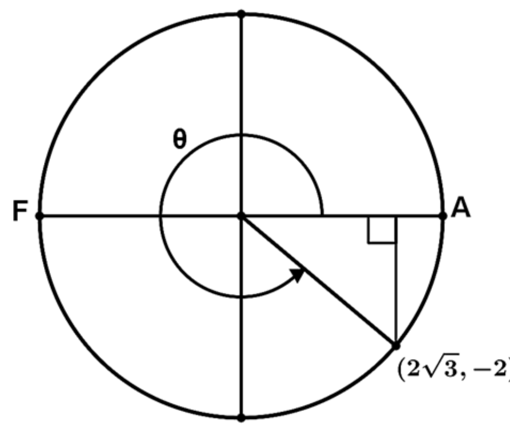
2. $\tan\theta =$



3. $\tan\theta =$



4. $\tan\theta =$



5. In each graph above, the angle of rotation is indicated by an arc and θ. Describe the angles of rotation from 0 to 2π that make tangent be positive and the angles of rotation that make tangent be negative.

Set

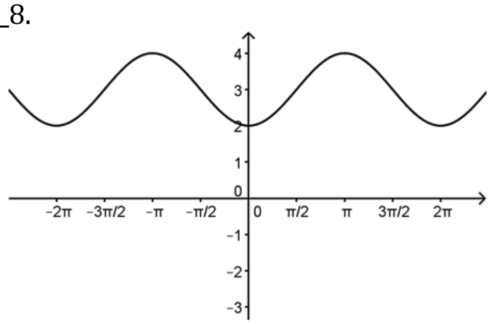
Topic: Transformations of trigonometric graphs

Match each trigonometric representation on the left with an equivalent representation on the right. Then check your answers with a graphing utility. Record your answer in the space provided to the left of the question number.

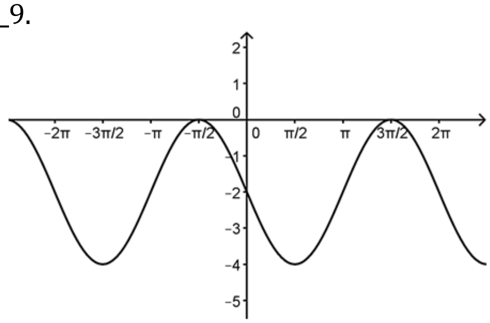
____6. $y = -3\sin\left(\theta + \frac{\pi}{2}\right)$

____7. $y = 3\cos\left(\theta + \frac{\pi}{2}\right)$

____8.



____9.



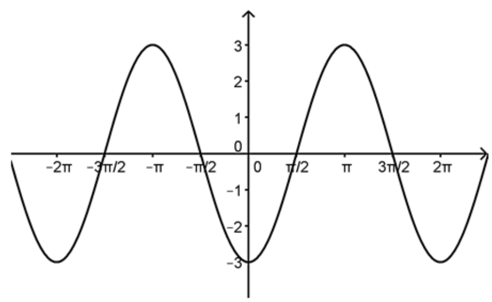
____10. $y = \sin\left(2\left(\theta + \frac{\pi}{2}\right)\right) - 2$

____11. $y = \sin(x + \pi)$

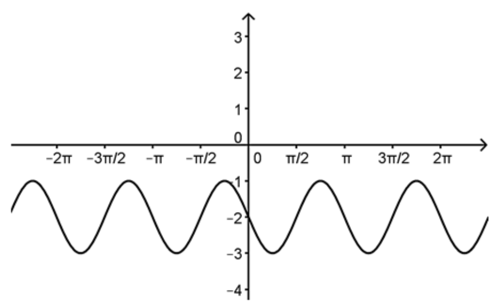
A. $y = -3\sin\theta$

B. $y = -\sin\theta$

C.



D.



E. $y = 2\cos\left(\theta + \frac{\pi}{2}\right) - 2$

F. $y = \cos(x + \pi) + 3$

12. Choose the equation(s) at the right that has the same graph as $y = \cos\theta$.

a. $y = \cos(-\theta)$
b. $y = \cos(\theta - \pi)$

Use the unit circle to explain why they are the same.

13. Choose the equation(s) at the right that has the same graph as $y = -\sin\theta$.

a. $y = \sin(\theta + \pi)$
b. $y = \sin(\theta - \pi)$

Use the unit circle to explain why they are the same.

For each function, identify the amplitude, period, horizontal shift, and vertical shift. Then sketch a graph.

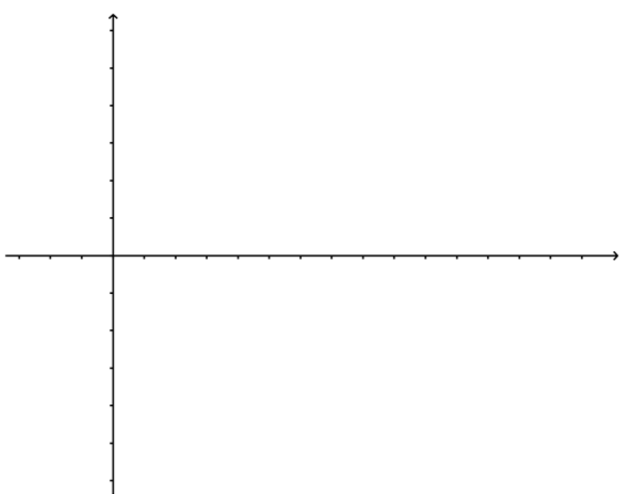
14. $f(t) = 14\cos\left(\frac{\pi}{6}(t-8)\right) + 8$

amplitude:

period:

horizontal shift:

vertical shift:



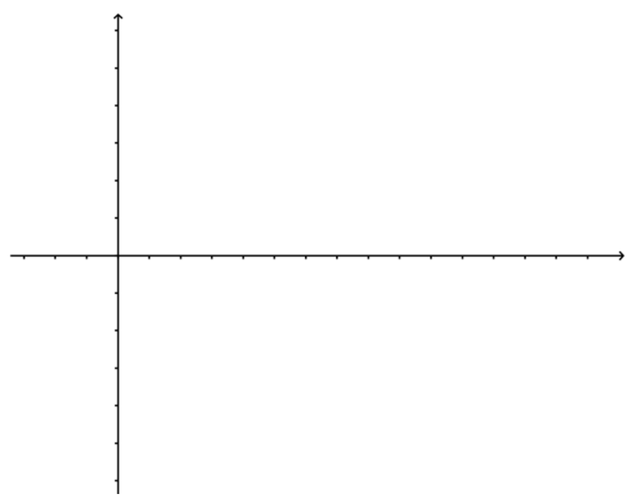
15. $f(t) = 4.5\sin\left(\frac{\pi}{4}t + \frac{3}{4}\right) + 8$

amplitude:

period:

horizontal shift:

vertical shift:



Go

Topic: Composite trigonometric functions

Recall that a composite function places one function such as $g(x)$*, inside the other,* $f(x)$*, by replacing the x in* $f(x)$ *with the entire function* $g(x)$*. In general, the notation is* $f\left(g(x)\right)$*. It is possible to do composition of the trigonometric functions. The answer to* $\sin^{-1}\left(\frac{1}{2}\right)$ *is an angle of* $30°$ *. The composition of* $\sin\left(\sin^{-1}\left(\frac{1}{2}\right)\right)$ *is simply asking "What is value of* $\sin 30°$*?" The answer is* $\frac{1}{2}$*.*

Sine was just "undoing" what $\sin^{-1}\theta$ *was doing.* ***Not all*** *composite trigonometric functions are inverses such as problems 19 – 24.*

Answer the following. For questions 19-24, it may help to draw a diagram.

16. $\sin\left(\sin^{-1}\frac{\sqrt{2}}{2}\right)$

17. $\cos\left(\cos^{-1}\frac{\sqrt{3}}{2}\right)$

18. $\tan\left(\tan^{-1}\frac{2}{3}\right)$

19. $\sin\left(\cos^{-1}\frac{1}{2}\right)$

20. $\cos(\tan^{-1} 1)$

21. $\sin\left(\tan^{-1}\frac{11}{4}\right)$

22. $\cos(\sin^{-1} 1)$

23. $\cos(\tan^{-1} 0)$

24. $\sin(\cot^{-1} 0)$

Name **Trigonometric Functions | 6.15**

Ready, Set, Go!

Ready

Topic: Rigid and non-rigid transformations of functions

The equation of a parent function is given. Write a new equation with the given transformations. Then sketch the new function on the same graph as the parent function. If the function has asymptotes, sketch them in.

1. $y = x^2$

   Vertical Shift: down 8

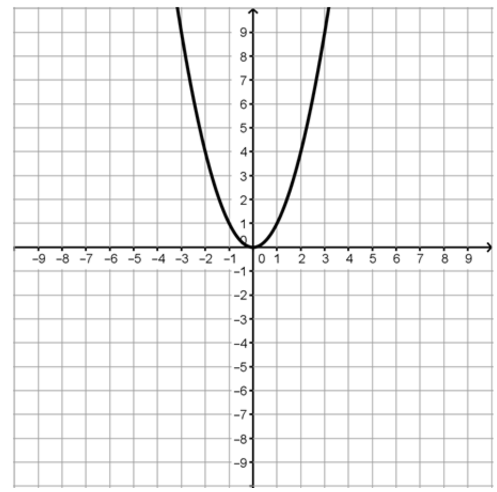
   Horizontal Shift: left 6

   Dilation: $\frac{1}{4}$

   Equation:

   Domain:

   Range:



2. $y = \frac{1}{x}$

   Vertical Shift: up 3

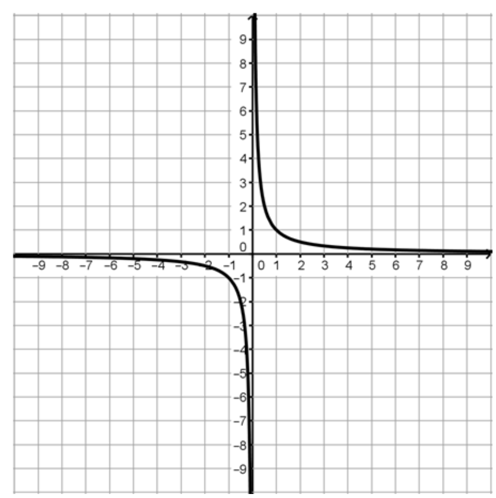
   Horizontal Shift: left 4

   Dilation: -1

   Equation:

   Domain:

   Range:



3. $y = \sqrt{x}$
Vertical Shift: none

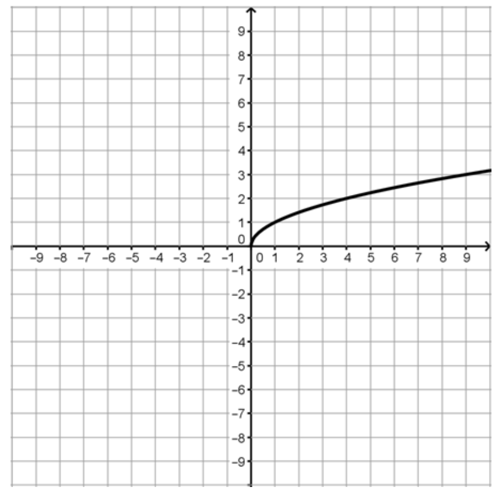
Horizontal Shift: left 5

Dilation: 3

Equation:

Domain:

Range:



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4. $y = \sin x$
Vertical Shift: up 1

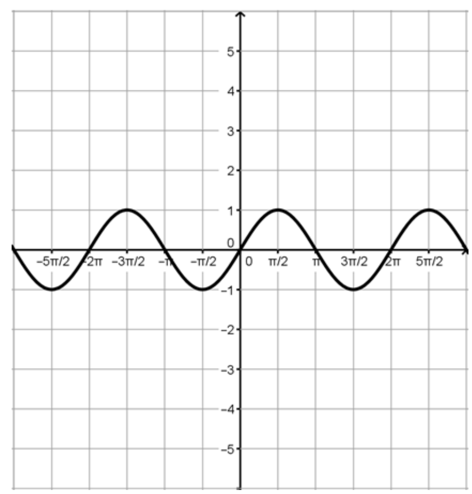
Horizontal Shift: left $\frac{\pi}{2}$

Dilation (amplitude): 3

Equation:

Domain:

Range:

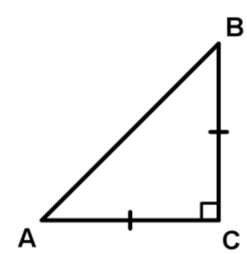


Set

Topic: Features of the graphs of the trigonometric functions

5. ΔABC is a right triangle with $AB = 1$.

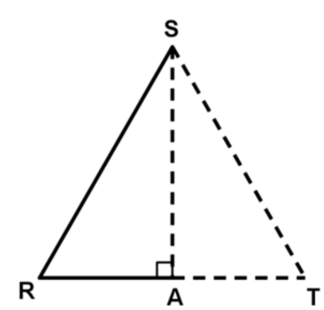
   Use the information in the figure to label the length of the sides and measure of the angles.



6. ΔRST is an equilateral triangle.
   $RS = 1$ and $\overline{SA}$ is an altitude

   Use the information in the figure to label the length of the sides, RA, and the *exact* length of $\overline{SA}$.

   Label the measure of $\angle RSA$ and $\angle SRA$.



7. Use the information from the figures in questions 6 and 7 to fill in the table.

Function	$\boldsymbol{\theta = 30°}$	$\boldsymbol{\theta = \frac{\pi}{6}}$	$\boldsymbol{\theta = 45°}$	$\boldsymbol{\theta = \frac{\pi}{4}}$	$\boldsymbol{\theta = 60°}$	$\boldsymbol{\theta = \frac{\pi}{3}}$
$\boldsymbol{\sin\theta}$						
$\boldsymbol{\cos\theta}$						
$\boldsymbol{\tan\theta}$						

8. Name the angles of rotation, in radians, where sine equals 0 in the domain $[0, 2\pi]$.

9. Name the angles of rotation, in radians, where cosine equals 0 in the domain $[0, 2\pi]$.

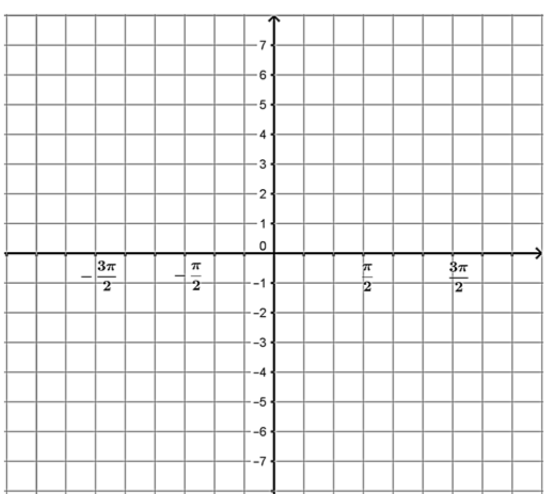
10. Name the angles of rotation, in radians, where tangent equals 0 in the domain $[0, 2\pi]$.

11. Name the angles of rotation, in radians, where tangent is undefined in the domain $[0, 2\pi]$.

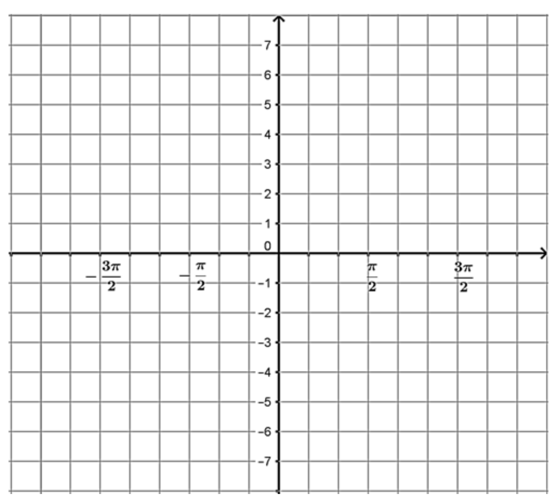
Topic: Graphing tangent functions.

Graph each tangent function. Be sure to indicate the location of the asymptotes.

12. $y = 3\tan(2\theta) - 4$



13. $y = 2\tan\left(\frac{1}{2}\theta\right) + 2$



Go

Topic: Trigonometric facts

Answer the questions below. Be sure you can justify your thinking.

14. Given triangle ABC with $\angle C$ being the right angle, what is $m\angle A + m\angle B$?

15. Identify the quadrants in which $\sin\theta$ is positive.

16. Identify the quadrants in which $\cos\theta$ is negative.

17. Identify the quadrants in which $\tan\theta$ is positive.

18. Explain why it is impossible for $\sin\theta > 1$.

19. Name the angles of rotation, in radians, for when $\sin\theta = \cos\theta$.

20. Which trigonometric function has the same value when the angle of rotation is positive or negative?

21. Write the Pythagorean identity and then prove it.

22. Explain why, in the unit circle, $\tan\theta = \frac{y}{x}$.

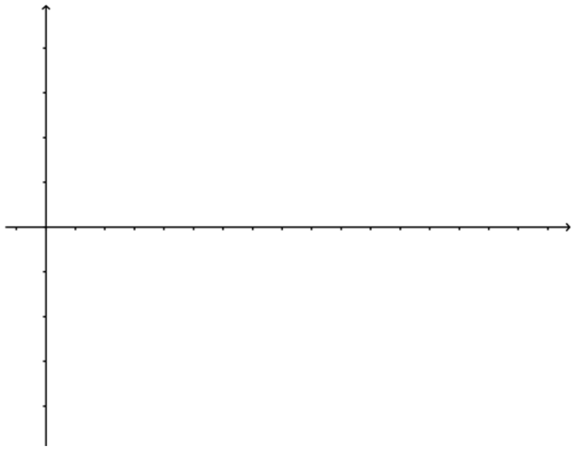
23. Which function represents the slope of the hypotenuse in a right triangle?

24. Name the trigonometric function(s) that are odd functions.

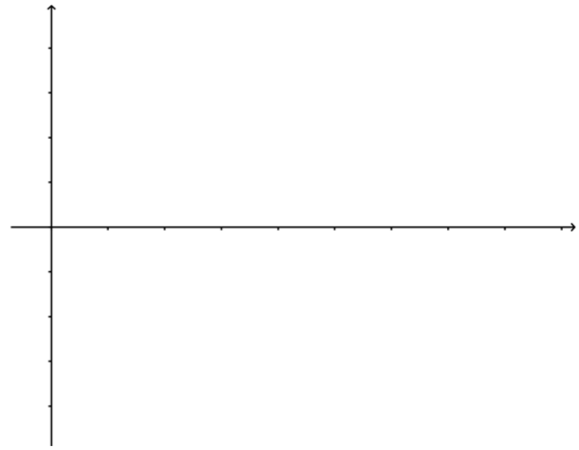
Topic: Graphing sine and cosine

Sketch the graph of each function. Include two full periods. Label the scale of your horizontal axis.

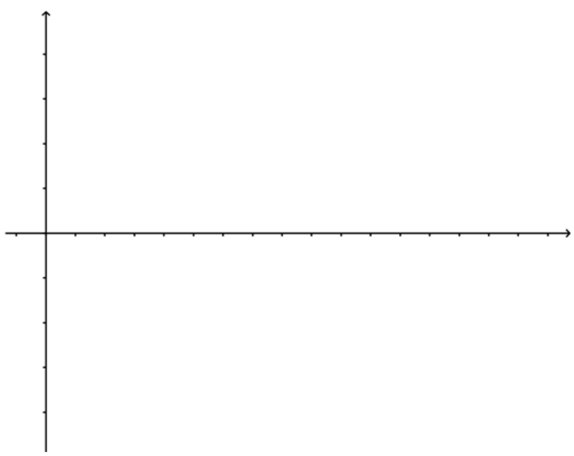
25. $y = 3\sin\left(\frac{\pi}{4}x\right)$



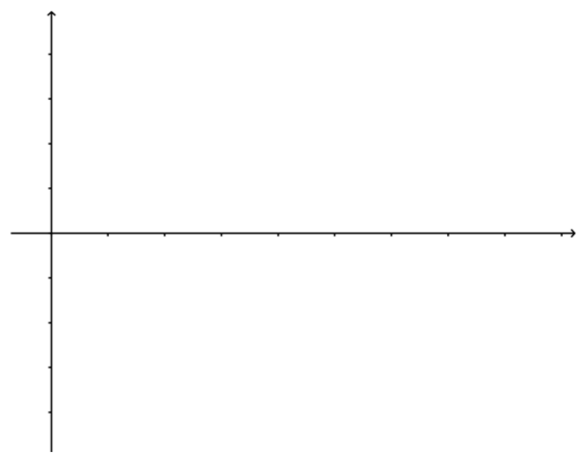
26. $y = \cos\left(x - \frac{\pi}{2}\right) + 3$



27. $y = 2\sin(3x) - 1$



28. $y = 4\cos\left(\frac{\pi}{8}x\right) - 5$



Name: **Trigonometric Functions | 6.16**

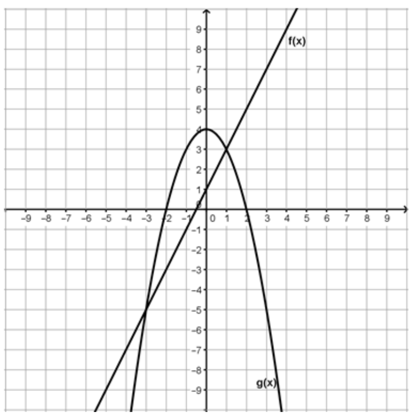
Ready, Set, Go!

Ready

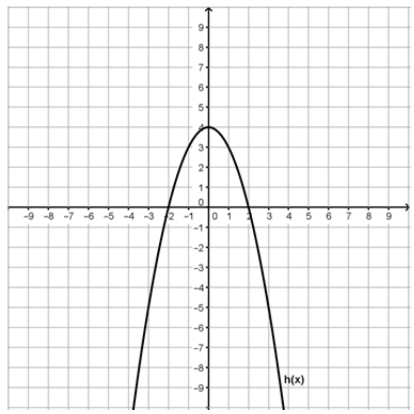
Topic: Function combinations

The functions $f(x)$, $g(x)$, and $h(x)$ are given in the graphs below. Graph the indicated combination on the same axes.

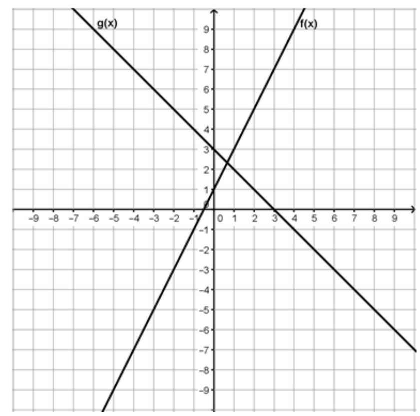
1. $f(x) + h(x)$



2. $\frac{1}{h(x)}$



3. $f(x)g(x)$



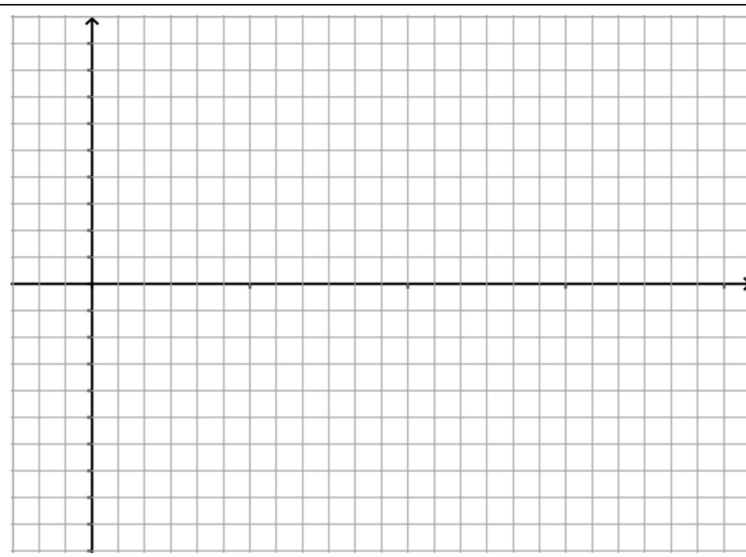
Set

Topic: Graphing reciprocal trigonometric functions

Graph each function. Remember, it may be helpful to sketch sine, cosine, or tangent first. Use the space below the function to construct a table of values to use for your graph.

4. $y = 3\csc(x)$

x	$3\sin x$	$3\csc x$



5. $y = 1 + \sec(x)$

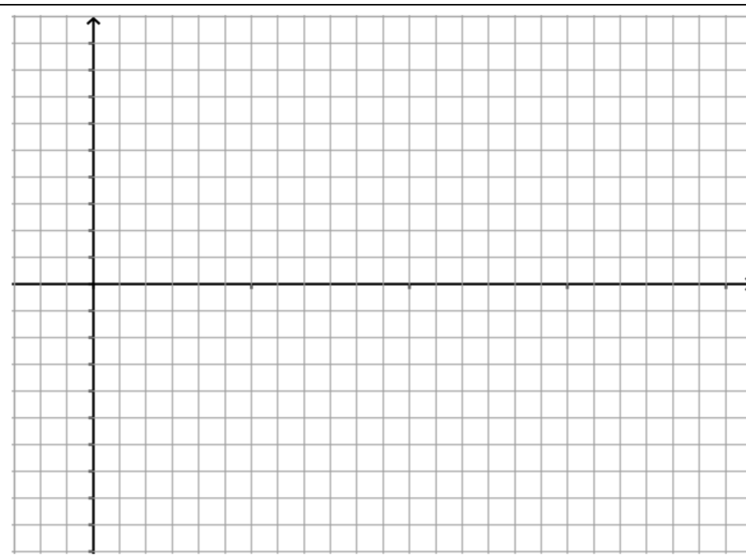
x	$1 + \cos x$	$1 + \sec x$

6. $y = \csc(2x)$

x	$\sin(2x)$	$\csc(2x)$

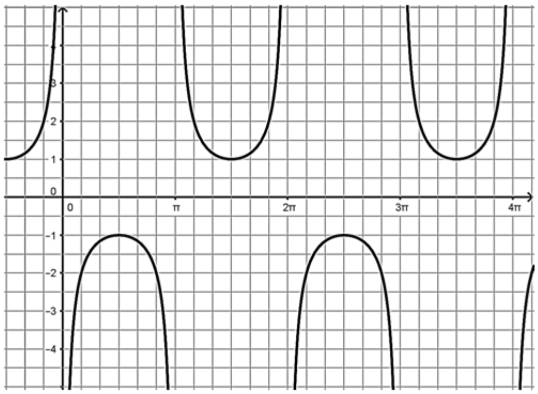
7. $y = \cot(2x)$

x	$\tan(2x)$	$\cot(2x)$

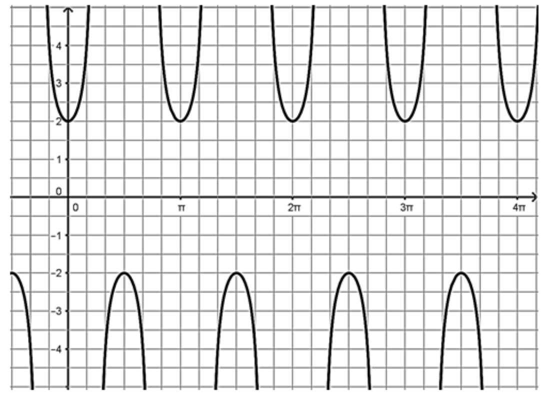


Write the equation of the given graphs. Check the accuracy of your equations using a graphing utility.

8. Use $\csc x$



9. Use $\sec x$



Go

Topic: Finding all six trigonometric ratios

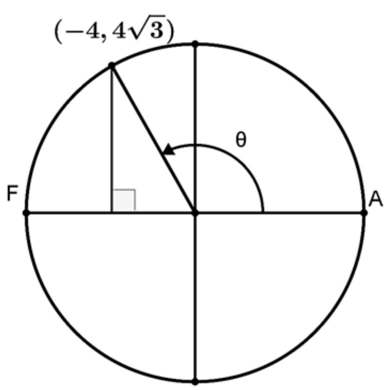
Use the given information to find the exact value of all remaining trigonometric ratios of θ. Hint: It may be helpful to draw a diagram.

10. $\sin\theta = \frac{3}{5}, \sec\theta < 0$

11. $\sec\theta = \frac{7}{2}, \tan\theta > 0$

12. $\cot\theta = \frac{4}{3}, \pi < \theta < \frac{\pi}{2}$

13.



Integrated Math 3
Module 7
Modeling with Functions

Adapted from

The Mathematics Vision Project:
Scott Hendrickson, Joleigh Honey, Barbara Kuehl,
Travis Lemon, Janet Sutorius

Module 7 Overview

Prerequisite Concepts & Skills:
- Identifying translations on families of functions
- Identifying features of functions
- Graphing families of functions (including sinusoidal)
- Understanding the effects of parameters on the graphs of functions
- Arithmetic on functions
- Evaluating functions
- Using multiple representations to interpret functions and to write function equations

Summary of the Concepts & Skills in Module 7:
- Using features of families of functions to write rules for geometric & function notation for given translations.
- Predicting the shapes of sums and products of functions
- Verifying the shapes of sums and products of functions using a graphing utility
- Describing features of curves formed by the sums and products of functions
- Writing dampened sinusoidal functions given a graph and table of values
- Composing functions within a context
- Interpreting the meaning of compositions of functions within a context
- Creating compositions of functions from a set of functions
- Decomposing compositions of functions given a set of functions and/or a table of values
- Combining functions using graphical, numeric, and algebraic representations

Content Standards and Standards of Mathematical Practice Covered:
- Content Standards: F.BF.1, F.BF.3, G.CO.2
- Standards of Mathematical Practice:
    1. Make sense of problems & persevere in solving them
    2. Reason abstractly & quantitatively
    3. Construct viable arguments & critique the reasoning of others
    4. Model with mathematics
    5. Use appropriate tools strategically
    6. Attend to precision
    7. Look for & make use of structure
    8. Look for & express regularity in repeated reasoning

Module 7 Vocabulary:
- Family of functions
- Parent function
- Geometric notation
- Function notation
- Pre-image
- Image
- Composition of functions
- Oscillation
- Decay
- Midline
- Dampened sinusoidal function
- Decompose
- Component parts of function compositions
- Function machine

Concepts Used in the Next Module:
The concepts covered in this module do not directly relate to the concepts covered in Module 8. The concepts covered in Module 8 include:
- Standard deviation
- Normal distributions
- z-score
- Methods of sampling
- Difference between survey, observational studies, and experiments
- Using simulations to estimate the likelihood of an event

Module 7 – Modeling with Functions

7.1 *Examining the transformations of a variety of familiar functions using tables* **(F.BF.3, G.CO.2)**
Warm Up: Discuss *Shifty Functions – A Solidify Understanding Task* (homework)
Classroom Task: Function Family Reunion – A Solidify Understanding Task
Ready, Set, Go Homework: Modeling with Functions 7.1

7.2 *Predicting the shape of a graph that is the sum or product of familiar functions* **(F.BF.1b)**
Warm Up: Combining Functions
Classroom Task: Imagineering – A Develop Understanding Task
Ready, Set, Go Homework: Modeling with Functions 7.2

7.3 *Combining a variety of functions using arithmetic operations to model complex behavior* **(F.BF.1b)**
Warm Up: Graphing Combined Functions
Classroom Task: The Bungee Jump Simulator – A Solidify Understanding Task
Ready, Set, Go Homework: Modeling with Functions 7.3

7.4 *Combining a variety of functions using function composition to model complex behavior* **(F.BF.1c+)**
Warm Up: Rumble Grumble
Classroom Task: Composing and Decomposing – A Develop Understanding Task
Ready, Set, Go Homework: Modeling with Functions 7.4

7.5 *Examining function transformations by composing and decomposing functions* **(F.BF.1c+, F.BF.3)**
Warm Up: Composing the Numbers
Classroom Task: Translating My Composition – A Solidify Understanding Task
Ready, Set, Go Homework: Modeling with Functions 7.5

7.6 *Combining functions defined by tables, graphs or equations using function composition and/or arithmetic operations* **(F.BF.1b, F.BF.1c+)**
Warm Up: Functions Galore
Classroom Task: Different Combinations – A Practice Understanding Task
Ready, Set, Go Homework: Modeling with Functions 7.6

7.1 Function Family Reunion
A Solidify Understanding Task

During the past few years of math classes, you have studied a variety of functions: linear, exponential, quadratic, polynomial, rational, radical, absolute value, logarithmic, and trigonometric.

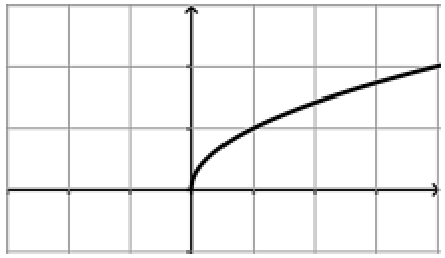
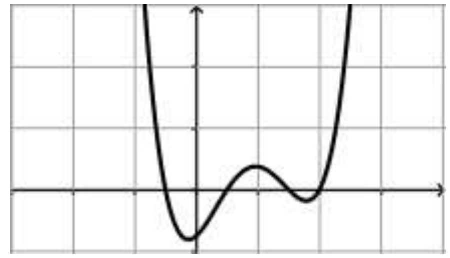
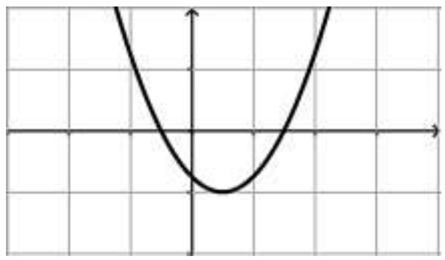
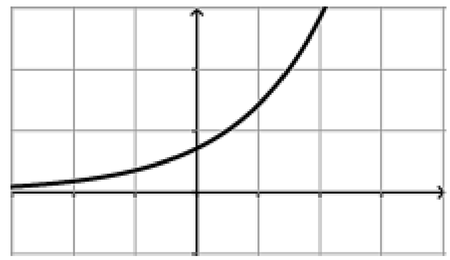
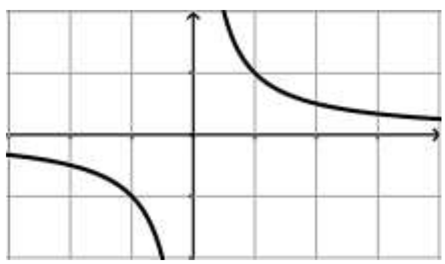
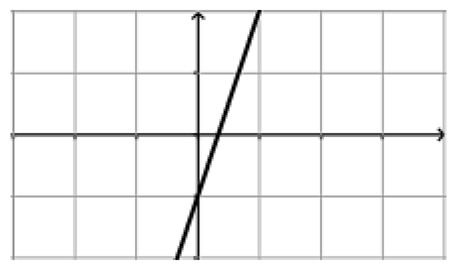
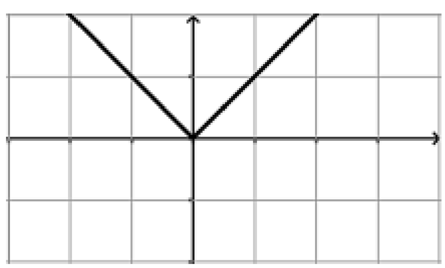
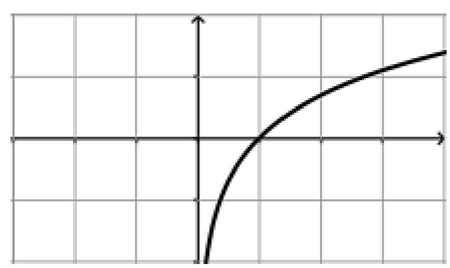
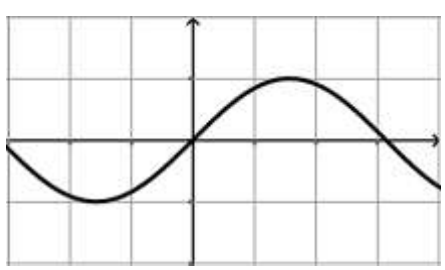
Like a family, each of these types of functions have similar characteristics that differ from other types of functions, making them uniquely qualified to model specific types of real world situations. Because of this, sometimes we refer to each type of function as a **"family of functions."**

1. Match each function family with the algebraic notation that best defines it.

Function Family Name	Algebraic Notation of the Parent Function
____1. linear	A. $y = \|x\|$
____2. exponential	B. $y = a\sin(bx)$ or $y = a\cos(bx)$ or $y = a\tan(bx)$ $y = a\csc(bx)$ or $y = a\sec(bx)$ or $y = a\cot(bx)$
____3. quadratic	C. $y = mx + b$
____4. polynomial	D. $y = \log x$
____5. rational	E. $y = ax^2 + bx + c$
____6. absolute value	F. $y = \frac{1}{x}$
____7. logarithmic	G. $y = ab^x$
____8. trigonometric	H. $y = a_n x^n + a_{n-1}x^{x-1} + a_{n-2}x^{n-2} + \cdots + a_0$
____9. radical	I. $y = \sqrt[n]{x}$

Just like your family, each member of a function family resembles other members of the family, but each has unique differences, such as being "wider" or "skinnier", "taller" or "shorter", or other features that allow us to tell them apart. We might say that each family of functions has a particular **"genetic code"** that gives its graph its characteristic shape. We might refer to the simplest form of a particular family as **"the parent function"** and consider all transformations of this parent function to be **"children"** within the same family.

2. Match each function family with the characteristic shape of the graph that fits it.

Function Family Name	**Characteristic Shape of the Parent Function**
____1. linear ____2. exponential	A.  B. 
____3. quadratic ____4. polynomial	C.  D. 
____5. rational ____6. absolute value	E.  F. 
____7. logarithmic ____8. trigonometric	G.  H. 
____9. radical	I. 

Function family characteristics are passed on to their "children" through a variety of transformations. While the members of each family shares common characteristics, transformations make each member of a family uniquely qualified to accomplish the mathematical work they are required to do.

3. For each of the following tables, a set of coordinate points that captures the characteristics of a parent graph is given. The additional columns give coordinate points for additional "children" of the family after a particular transformation has occurred. **Write the rule for each of the different transformations of the parent graph**. Note: We can think of each new set of coordinate points (that is, the *image* points) as a geometric transformation of the original set of coordinate points (that is, the *pre-image* points) and use the notation associated with geometric transformations to describe transformation. Or, we can write the rule using algebraic function notation. **Use both types of notation to represent each transformation.** As you work on each problem, think about how you can check that the function is correct.

	Pre-image (parent graph)	**Image 1**	**Image 2**	**Image 3**
Geometric notation	(x, y)	$(x, y) \rightarrow (x, y + 2)$		
Function notation	$f(x) = x^2$	$f_1(x) = x^2 + 2$		
Selected points that fit this image	$(-2, 4)$	$(-2, 6)$	$(-2, 8)$	$(-3, 4)$
	$(-1, 1)$	$(-1, 3)$	$(-1, 2)$	$(-2, 1)$
	$(0, 0)$	$(0, 2)$	$(0, 0)$	$(-1, 0)$
	$(1, 1)$	$(1, 3)$	$(1, 2)$	$(0, 1)$
	$(2, 4)$	$(2, 6)$	$(2, 8)$	$(1, 4)$

	Pre-image (parent graph)	**Image 1**	**Image 2**	**Image 3**
Geometric notation	(x, y)			
Function notation	$f(x) = 2^x$			
Selected points that fit this image	$\left(-2, \frac{1}{4}\right)$	$(-2, 1)$	$\left(-2, -\frac{1}{4}\right)$	$\left(-3, \frac{1}{4}\right)$
	$\left(-1, \frac{1}{2}\right)$	$(-1, 2)$	$\left(-1, -\frac{1}{2}\right)$	$\left(-2, \frac{1}{2}\right)$
	$(0, 1)$	$(0, 4)$	$(0, -1)$	$(-1, 1)$
	$(1, 2)$	$(1, 8)$	$(1, -2)$	$(0, 2)$
	$(2, 4)$	$(2, 16)$	$(2, -4)$	$(1, 4)$

	Pre-image (parent graph)	Image 1	Image 2	Image 3
Geometric notation	(x, y)			
Function notation	$f(x) = \lvert x \rvert$			
Selected points that fit this image	$(-2, 2)$	$(-2, -4)$	$(2, 2)$	$(-5, 2)$
	$(-1, 1)$	$(-1, -2)$	$(3, 1)$	$(-4, 1)$
	$(0, 0)$	$(0, 0)$	$(4, 0)$	$(-3, 0)$
	$(1, 1)$	$(1, -2)$	$(5, 1)$	$(-2, 1)$
	$(2, 2)$	$(2, -4)$	$(6, 2)$	$(-1,2)$

	Pre-image (parent graph)	Image 1	Image 2	Image 3
Geometric notation	(x, y)			
Function notation	$f(x) = \sin x$			
Selected points that fit this image	$(0, 0)$	$(0, 2)$	$(0, 0)$	$(0, 0)$
	$\left(\frac{\pi}{2}, 1\right)$	$\left(\frac{\pi}{2}, 3\right)$	$\left(\frac{\pi}{4}, 1\right)$	$\left(\frac{\pi}{2}, -2\right)$
	$(\pi, 0)$	$(\pi, 2)$	$\left(\frac{\pi}{2}, 0\right)$	$(\pi, 0)$
	$\left(\frac{3\pi}{2}, -1\right)$	$\left(\frac{3\pi}{2}, 1\right)$	$\left(\frac{3\pi}{4}, -1\right)$	$\left(\frac{3\pi}{2}, 2\right)$
	$(2\pi, 0)$	$(2\pi, 2)$	$(\pi, 0)$	$(2\pi, 0)$

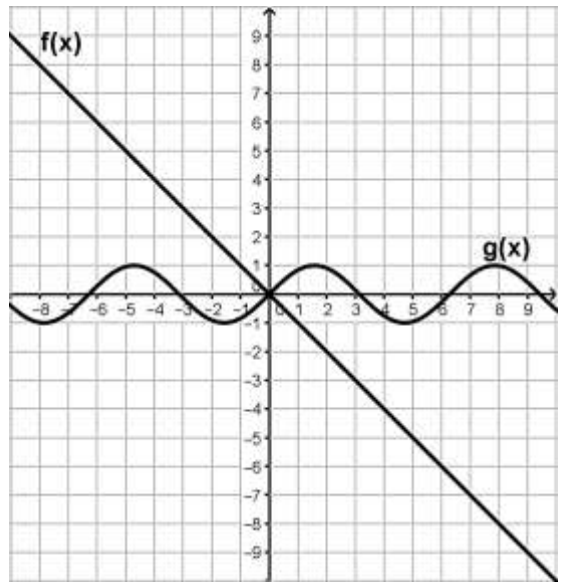
	Pre-image (parent graph)	Image 1	Image 2	Image 3
Geometric notation	(x, y)			
Function notation	$f(x) = \sqrt{x}$			
Selected points that fit this image	$(0, 0)$	$(0, 0)$	$(0, 0)$	$(3, 0)$
	$(1, 1)$	$\left(1, \frac{1}{2}\right)$	$\left(\frac{1}{2}, 1\right)$	$(4, 1)$
	$(4, 2)$	$(4, 1)$	$(2, 2)$	$(7, 2)$
	$(9, 3)$	$\left(9, \frac{3}{2}\right)$	$\left(\frac{9}{2}, 3\right)$	$(12, 3)$
	$(16, 4)$	$(16, 2)$	$(8, 4)$	$(19, 4)$

7.2 Warm Up
Combining Functions

The output values of $f(x)$ and $g(x)$ are provided in the table below. Use these output values to find the values for the combined function, $h(x)$. Then graph the combined function, $h(x)$.

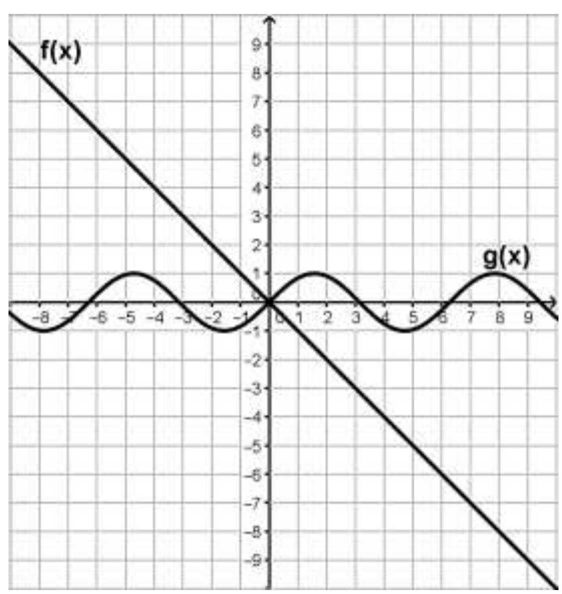
1. Use the graphs below for $f(x) = -x$ and $g(x) = \sin x$ to graph the combined function, $h(x) = f(x) + g(x)$.

x	$f(x)$	$g(x)$	$h(x)$
-2π	6.28	0	
$-\frac{3\pi}{2}$	4.71	1	
$-\pi$	3.14	0	
$-\frac{\pi}{2}$	1.57	-1	
0	0	0	
$\frac{\pi}{2}$	-1.57	1	
π	-3.14	0	
$\frac{3\pi}{2}$	-4.71	-1	
2π	-6.28	0	



2. Use the graphs below for $f(x) = -x$ and $g(x) = \sin x$ to graph the combined function, $h(x) = f(x) \cdot g(x)$.

x	$f(x)$	$g(x)$	$h(x)$
-2π	6.28	0	
$-\frac{3\pi}{2}$	4.71	1	
$-\pi$	3.14	0	
$-\frac{\pi}{2}$	1.57	-1	
0	0	0	
$\frac{\pi}{2}$	-1.57	1	
π	-3.14	0	
$\frac{3\pi}{2}$	-4.71	-1	
2π	-6.28	0	



7.2 Imagineering
A Develop Understanding Task

Part I:

You are excited to get to vote on the plans for a proposed new thrill ride at a local theme park. The engineers want public input on the design for the new ride. You are one of ten teenagers who have been selected to review the plans based on your good math grades!

As your excitement mounts, the engineers begin their presentation. To your dismay, there are no models or illustrations of the proposed rides; each ride is described only with equations. The equations represent the path a rider would follow through the course of the ride.

Unfortunately, your cell phone, which contains a graphing calculator app, is completely dead due to too much texting and surfing the internet. Therefore, you are trying hard to keep up with the presentation by imagining what the graphs of each of these equations would look like. While each equation consists of functions you are familiar with, the combination of functions in each equation has you wondering about their combined effects.

For each of the following proposed thrill rides, use your imagination and best reasoning about the individual functions involved to **sketch** a graph of the path of the rider. Let *y* represent the height of the rider above the ground and let *x* represent the distance from the start of the ride. Explain your reasoning about the shape of the graph. (Note: Use radians for trigonometric functions.)

Proposal #1: "*The Mountain Climb*"
The Equation: $y = 2x + 5\sin x$

My Predicted Graph:

My Explanation:

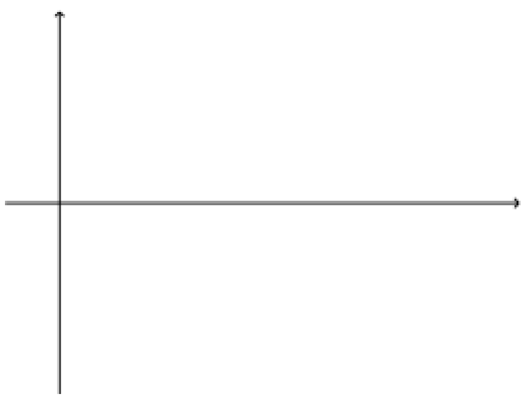
Proposal #2: "*The Periodic Bump*"
The Equation: $y = |10\sin x|$

My Predicted Graph:

My Explanation:

Proposal #3: "*The Amplifier*"
The Equation: $y = x \cdot \sin x$

My Predicted Graph:



My Explanation:

Proposal #4: "*The Gentle Wave*"
The Equation: $y = 10(0.9)^x \cdot \sin x$

My Predicted Graph:

My Explanation:

Proposal #5: "*The Spinning High Dive*"
The Equation: $y = (100 - 2x^2) + 5\sin(8x)$

My Predicted Graph:

My Explanation:

Part II:

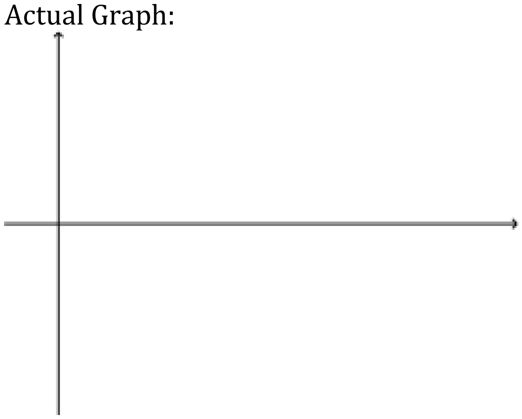
When you got home, your friends were all anxiously waiting to hear about the proposed new rides. After explaining the situation, your friends all pull out their calculators and they began comparing your imagined images with the actual graphs.

Some of your friends' graphs differed from the others because of their window settings. Some window settings revealed the features of the graphs you were expecting to see, while other window settings obscured those features.

Examine the actual graphs of each of the thrill ride proposals using a graphing utility. Select a window setting that will reveal as many of the features of the graphs as possible. Explain any differences between your imagined graphs and the actual graphs. What features did you get right? What features did you miss? Be sure to label the axes.

Proposal #1: "*The Mountain Climb*"
The Equation: $y = 2x + 5\sin x$

Actual Graph:

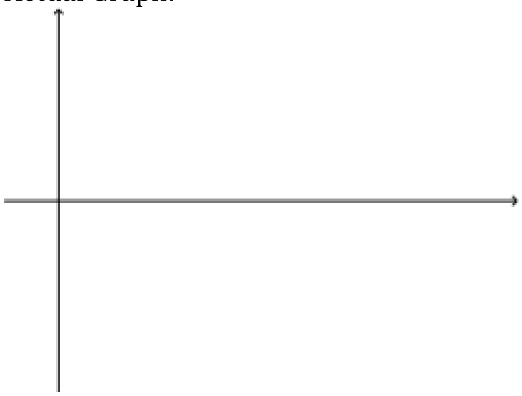


What features I got right:

What features I missed:

Proposal #2: "*The Periodic Bump*"
The Equation: $y = |10\sin x|$

Actual Graph:

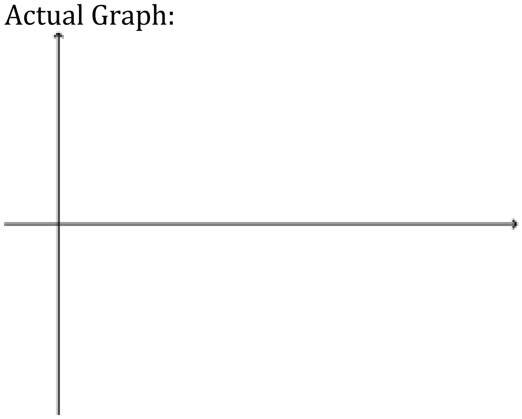


What features I got right:

What features I missed:

Proposal #3: "*The Amplifier*"
The Equation: $y = x \cdot \sin x$

Actual Graph:

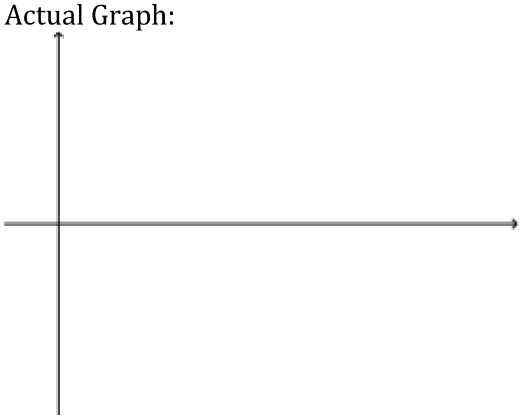


What features I got right:

What features I missed:

Proposal #4: "*The Gentle Wave*"
The Equation: $y = 10(0.9)^x \cdot \sin x$

Actual Graph:

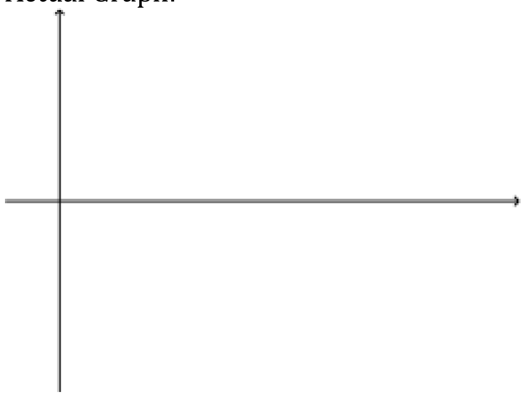


What features I got right:

What features I missed:

Proposal #5: "*The Spinning High Dive*"
The Equation: $y = (100 - 2x^2) + 5\sin(8x)$

Actual Graph:



What features I got right:

What features I missed:

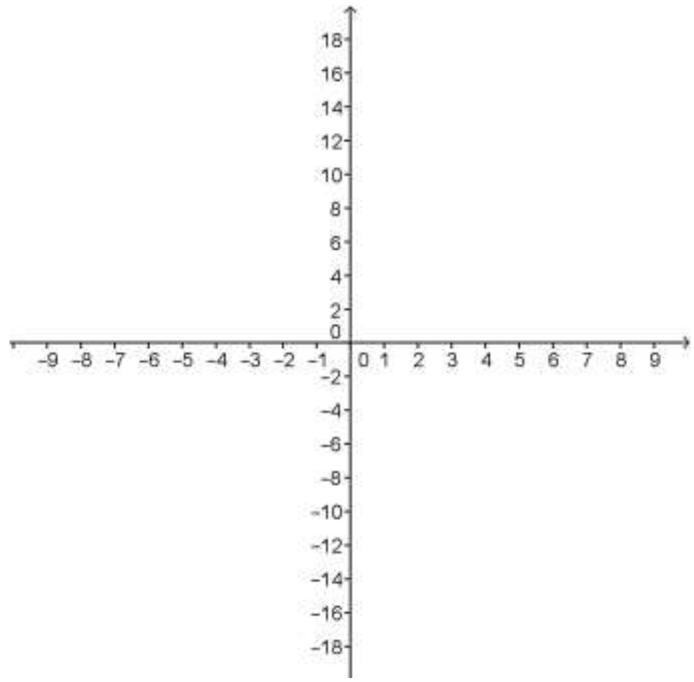
How would the graphs change if the function contained cosine instead of sine?

7.3 Warm Up
Graphing Combined Functions

Complete the table of values for the two "basic" functions and the combined function. Then use the table of values to sketch a graph of each function. Be sure to think about the features of the functions that are composed together to create the given functions.

1. $y = -2x \cdot \cos x$

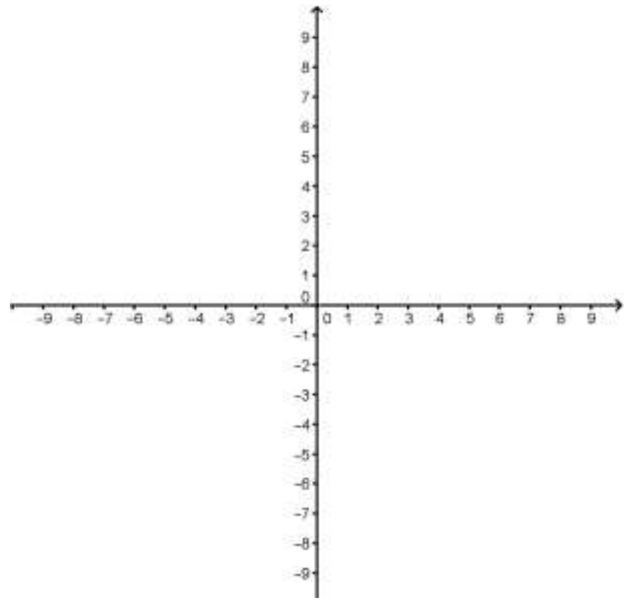
x	$-2x$	$\cos x$	$-2x \cdot \cos x$
-2π			
$-\frac{3\pi}{2}$			
$-\pi$			
$-\frac{\pi}{2}$			
$-\frac{\pi}{4}$			
0			
$\frac{\pi}{4}$			
$\frac{\pi}{2}$			
π			
$\frac{3\pi}{2}$			
2π			



How does multiplying two "basic" functions together impact the graph of the combined function?

2. $y = -\frac{1}{2}x + 3\cos x$

x	$-\frac{1}{2}x$	$3\cos x$	$-\frac{1}{2}x + 3\cos x$
-2π			
$-\frac{3\pi}{2}$			
$-\pi$			
$-\frac{\pi}{2}$			
0			
$\frac{\pi}{2}$			
π			
$\frac{3\pi}{2}$			
2π			



How does adding two "basic" functions together impact the graph of the combined function?

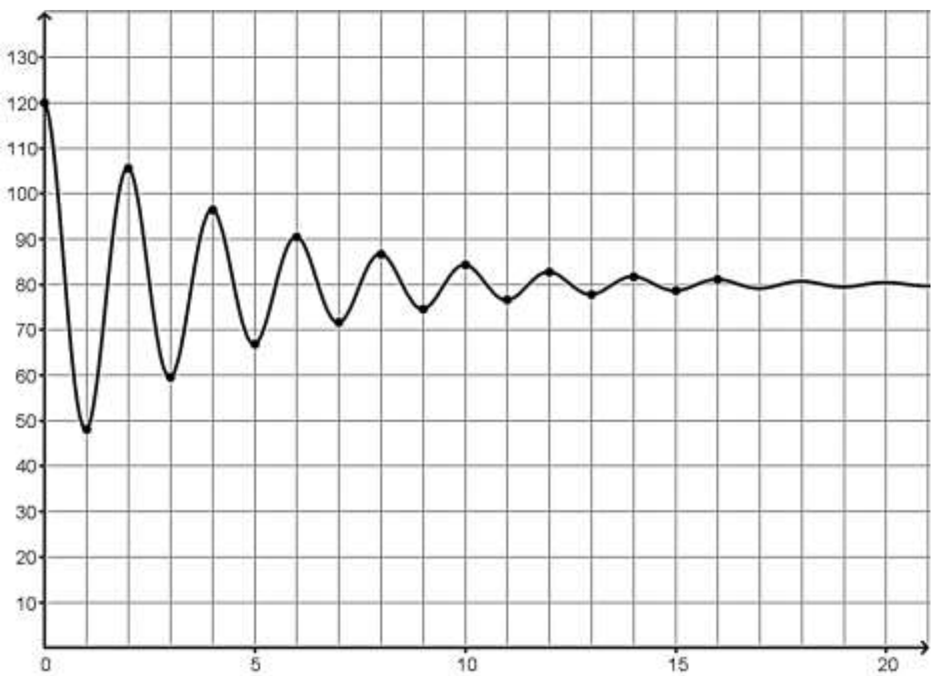
7.3 The Bungee Jump Simulator
A Solidify Understanding Task

As a reward for helping the engineers at the local amusement park select a design for their next ride, you and your friends get to visit the amusement park for free with one of the engineers as a tour guide. This time you remember to bring your graphing utility along, in case the engineers start to speak in "math equations" again.

Sure enough, just as you are about to get in line for the *Bungee Jump Simulator*, your guide pulls out a graph and begins to explain the mathematics of the ride. To prevent injury, the ride has been designed so that a bungee jumper follows the path given in this graph. The graph below, shows that jumpers are released from the top of the tower at the left, and dismount in the center of the tower at the right, after their up and down motion has stopped. The cable, to which their bungee cord is attached, moves the rider safely away from the left tower and allows for an easy exit at the right.

Your tour guide won't let you and your friends get in line for the ride until you have reproduced this graph on your graphing utility, exactly as it appears in this diagram.

1. Work with a partner to try and recreate this graph on your calculator screen. Make sure you pay attention to the height of the jumper at each oscillation, as given in the table.



Horiz.	Vert.	Distance from midline
0	120	40
1	48	32
2	105.6	25.6
3	59.52	20.48
4	96.38	16.38
5	66.89	13.11
6	90.49	10.49
7	71.61	8.39
8	86.71	6.71
9	74.63	5.37
10	84.30	4.30
11	76.56	3.44
12	82.75	2.75
13	77.80	2.20
14	81.76	1.76
15	78.59	1.41
16	81.13	1.13
…	…	

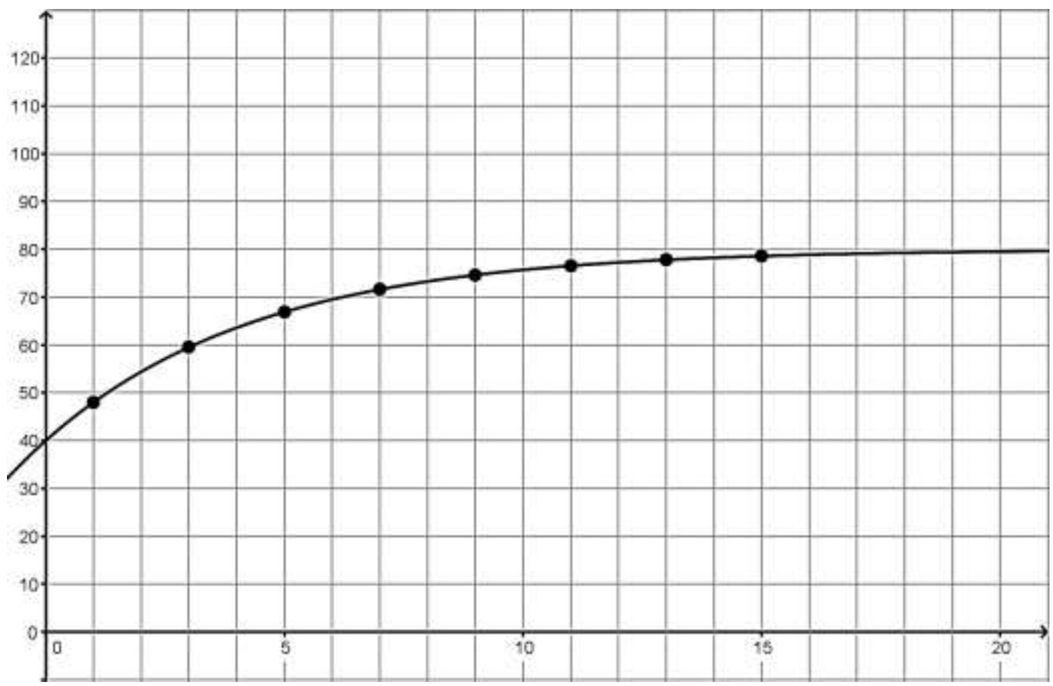
Record your equation of this graph here:

After a thrilling ride on the *Bungee Jump Simulator*, you are met by your guide who has a new puzzle for you. "As you are aware," says the engineer, "temperatures around here are very cold at night but very warm during the day. When designing rides, we have to take into account how the metal frames and cables might heat up throughout the day. Our calculations are based on Newton's Law of Heating. Newton found that while the temperature of a cold object increases when the air is warmer than the object, the rate of change of the temperature slows down as the temperature of the object gets closer to the temperature of its surrounding."

Of course the engineer has a graph of this situation, which he says "represents the decay of the difference between the temperature of the cables and the surrounding air."

Your friends think this graph reminds them of the points at the bottom of each of the oscillations of the bungee jump.

2. Using the clue given by the engineer, "This graph represents the *decay* of the difference between the temperature of the cables and the surrounding air," try to recreate this graph on your graphing utility. (Hint: What types of graphs do you generally think of when you are trying to model a growth or decay situation? What transformations might make such a graph look like this one?)

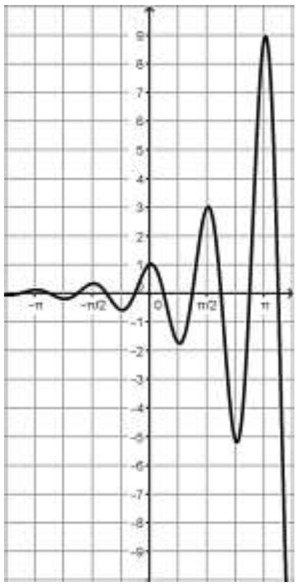


Record your equation of this graph here:

Practice Problems:

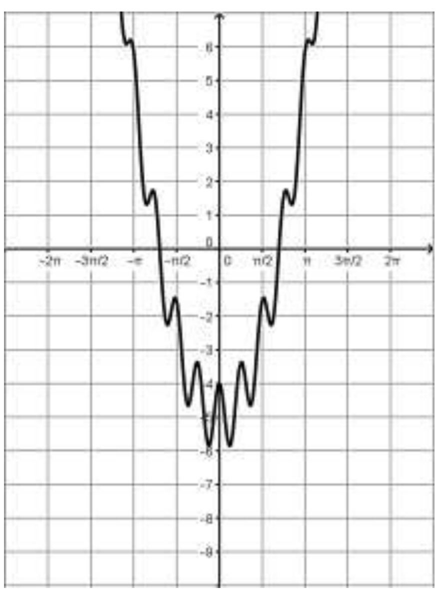
Write the equation of that model each of the following graphs.

1.

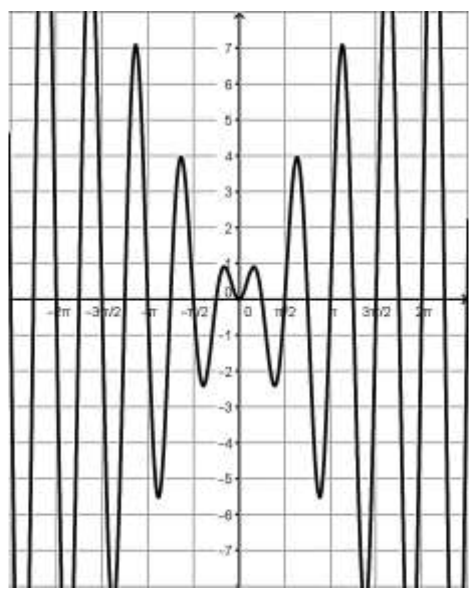


2.

3.



4.



7.4 Warm Up
Rumble, Grumble

When David is hungry, he gets grouchy. When he is not hungry, he stays in a good mood. In other words, David's mood is a function of his hunger level.

David's friend Karla usually has positive feelings toward him. When he is grouchy, however, Karla gets irritated with him. In other words, Karla's attitude toward David is a function of his mood.

Assume that David eats three meals a day, at normal times, so that his hunger level is a function of the time of day.

Sketch a graph of Karla's feelings toward David as a function of time over the course of two days.

7.4 Composing and Decomposing
A Develop Understanding Task

As the day gets warmer, you and your friends decide to cool off by taking a ride on the *Turbulent Waters Dive (TWD)*. As you are waiting in line your tour guide explains the mathematics behind designing the waiting area for a ride.

"As you can see," says the engineer, "the waiting area can be enlarged or reduced by moving a few chains around. The area we need for waiting guests depends on the time of day. We collect data for each ride so we can use functions to model the typical wait time and how much waiting area we need to provide for our guests."

And of course, your guide has the functions that represent this particular ride.

Average number of people in the *TWD* line as a function of time: $\boldsymbol{p(t) = 3000\cos\left(\frac{1}{5}(t-3)\right)}$

$\boldsymbol{t}$ is the number of hours before or after noon, so $t = 2$ represents 2:00 p.m. and $t = -2$ represents 10:00 a.m.
$\boldsymbol{p}$ represents the number of people in line

1. At what time of day is the number of people in line the largest?

2. What is the maximum number of people in line, based on the function for the average number of people in line?

3. When do you think the amusement park opens and closes, based on this function?

Waiting area required as a function of the number of people in line: $\boldsymbol{a(p) = 4p + 100}$

$\boldsymbol{a}$, the waiting area, is measured in square feet

4. In terms of the story context, what do you think the 4 and the 100 represent in function rule for waiting area?

Wait time for a guest as a function of the number of people in line: $\boldsymbol{w(p) = 60 \cdot \left(\frac{p-1500}{1500}\right)}$

$\boldsymbol{w}$, the wait time, is measured in minutes

5. In terms of the story context, what might be the meaning of the 1500 in the function rule for wait time? What is the domain of $w(p)$?

6. How much **waiting area** is required for the guests in line for the *Turbulent Waters Dive* at each of the times listed in the following table?

Time of Day	**Waiting Area Required (sq. ft.)**
10:00 a.m.	
12:00 noon	
2:00 p.m.	
4:00 p.m.	
8:00 p.m.	

a. For each time of day, you had to complete a series of calculations. Describe how you found the waiting area at different times.

b. Can you create a single rule that will determine the waiting area as a function of the time of day?

7. What is the **wait time** for a guest that arrives at the end of the line for the *Turbulent Waters Dive* at each of the times listed in the following table?

Time of Day	**Waiting Time (minutes)**
10:00 a.m.	
12:00 noon	
2:00 p.m.	
4:00 p.m.	
8:00 p.m.	

a. For each time of day, you had to complete a series of calculations. Describe how you found the wait time at different times of the day.

b. Can you create a single rule that will determine the wait time as a function of the time of day?

To maintain crowd control when the lines get long, cast members dressed as pirates (the *Turbulent Waters Dive* has a pirate theme), mingle with the waiting guests. Their antics distract the guests who listen attentively to their pirate jokes. The number of cast members needed depends on the number of people waiting in the line.

Number of cast members needed as a function of the number of people in line: $\boldsymbol{c(t) = \frac{p}{150}}$

$\boldsymbol{p}$ represents the number of people in line
$\boldsymbol{c}$ represents the number of cast members needed

8. In terms of the story context, what might be the meaning of the 150 in the function rule for cast members needed?

9. How many **cast members** are needed to entertain and distract the waiting guests at each of the following times of the day?

Time of Day	Cast Members Needed
10:00 a.m.	
12:00 noon	
2:00 p.m.	
4:00 p.m.	
8:00 p.m.	
t hours before or after noon ($t < 0$ before noon, $t > 0$ after noon)	

a. Write a single rule that will determine the number of cast members needed as a function of the time of day.

On warm, sunny days, misters are used to cool down the waiting guests. The number of misters that need to be turned on depends on the size of the waiting area that has been opened up to contain the number of people in line.

Number of misters needed as a function of the waiting area: $\boldsymbol{m(t) = \frac{a}{1000}}$

$\boldsymbol{a}$, the waiting area, is measured in square feet
$\boldsymbol{m}$ represents the number of misters to be turned on

10. In terms of the story context, what might be the meaning of the 1000 in the function rule for the number of misters needed?

11. How many **misters** need to be turned on to cool the waiting guests at each of the following times of day?

Time of Day	Misters Needed
10:00 a.m.	
12:00 noon	
2:00 p.m.	
4:00 p.m.	
8:00 p.m.	
t hours before or after noon ($t < 0$ before noon, $t > 0$ after noon)	

a. Write a single rule that will determine the number of misters needed as a function of the time of day.

12. Explain how the following diagram might help you think about the work you have been doing on the previous questions. How does the notation used in the diagram support the way you have been combining functions in this task? This way of combining functions is called ***function composition***.

$$\boldsymbol{x} \longrightarrow \boxed{\boldsymbol{g(x)}} \longrightarrow \boxed{\boldsymbol{f(x)}} \longrightarrow \boldsymbol{y = f(g(x))}$$

7.5 Warm Up
Composing the Numbers

Evaluate each composition using the following functions:

$f(x) = 3x - 4$ $\qquad$ $g(x) = 2x^2 - 1$ $\qquad$ $h(x) = \sqrt{x - 3}$

1. $f\left(g\big(h(12)\big)\right)$

2. $(h \circ g \circ f)(-2)$

3. $g\left(h\left(f\left(\frac{11}{3}\right)\right)\right)$

7.5 Translating My Composition
A Solidify Understanding Task

All this work with modeling rides and waiting lines at the local amusement park may have you wondering about the variety of ways of combining functions. In this task, we continue building new functions from familiar ones.

Suppose you have the following "starter set" of functions:

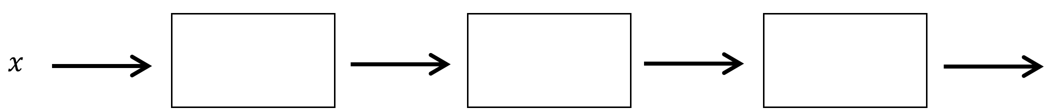
$f(x) = x + 5$
$g(x) = x^2$
$h(x) = 3x$
$j(x) = 2^x$
$k(x) = x - 1$

Example Problem: $c(x) = [3(x - 1)]^2$

a. List the operations being performed on x in the example function.

b. Use the flow diagram to show how three of the functions from the list above were composed (put together) in order to obtain the example function.

1. Your partner gives you $a(x) = 3(x + 5)^2$. List the operations being performed on x and then complete the diagram to determine the order of the functions that were composed to obtain $a(x)$.



2. To test your composition of functions, select one or two values for *x,* running them through your chain of function machines, to see if you get the same results as when you evaluate the given function rule for the same numbers. What do you notice when you do this?

3. Your partner gives you $b(x) = 2^{(3x)^2}$. List the operations being performed on x and then complete the diagram to determine the order of the functions that were composed to obtain $b(x)$.

$$x \longrightarrow \boxed{} \longrightarrow \boxed{} \longrightarrow \boxed{} \longrightarrow$$

4. Now it's your turn! Create your own function rule using the set of functions given at the beginning of this task and following the four steps given above. Your partner should do the same so you can exchange function rules.

   My function rule:

   My partner's function rule:

   List the operations being performed on x and then complete the diagram to determine the order of the functions that were composed to obtain your partner's function rule.

$$x \longrightarrow \boxed{} \longrightarrow \boxed{} \longrightarrow \boxed{} \longrightarrow$$

   Test your composition of functions for a few values. Make any adjustments necessary based on your test results.

5. Your partner thought he was being tricky and gave you the following composition of **all five** functions: $y = (2^{3x} - 1)^2 + 5$. Determine the composition of functions used to obtain the given function.

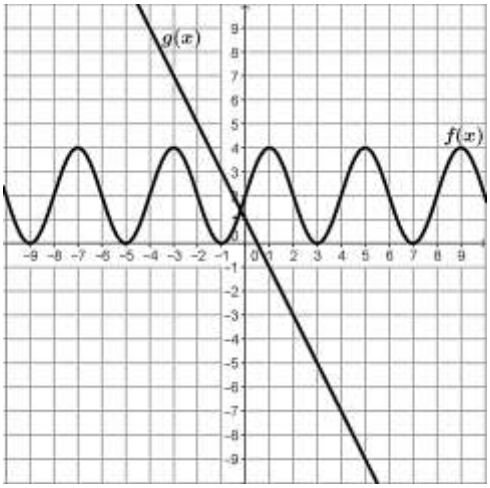
6. Given the functions $m(x) = x + 3$, $n(x) = \frac{1}{x}$, and $p(x) = x^3$, perform the following compositions:

   a. $p\left(n\big(m(x)\big)\right)$ b. $p\left(m\big(n(x)\big)\right)$ c. $m\left(p\big(m(x)\big)\right)$

7. Is function composition commutative? Give reasons to support your answer.

7.6 Warm Up
Functions Galore

Use the given graphs below to find the indicated values.



1. $k(x) = f\big(g(x)\big)$, find $k(1)$

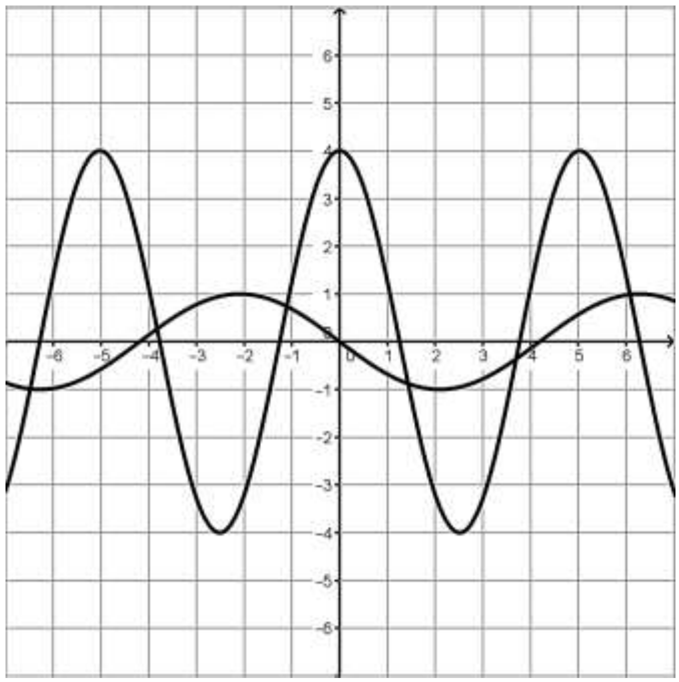
2. $h(x) = g\big(f(x)\big)$, find $h(5)$

3. $b(x) = g\big(g(x)\big) + f\big(f(x)\big)$, find $b(1)$

7.6 Different Combinations
A Practice Understanding Task

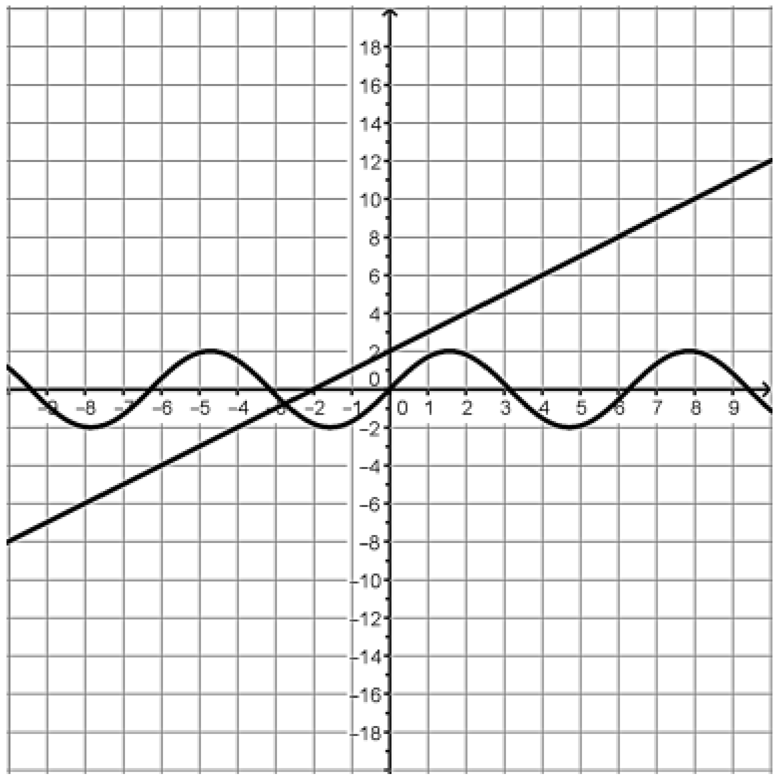
We have found the value of being able to combine different function types in various ways to model a variety of situations. In this task, you will practice combining functions when they are described in different ways: graphically, numerically, or algebraically.

1. **Add** the following two functions together graphically. See if you can produce the resulting graph by just working with the points on the two graphs and considering what happens when two functions are combined using the operation of addition.



What points are most helpful in determining the shape of the resulting graph, and why?

2. **Multiply** the following two functions together graphically. See if you can produce the resulting graph by just working with the points on the two graphs and considering what happens when two functions are combined using the operation of multiplication.



What points are most helpful in determining the shape of the resulting graph, and why?

3. Functions f and g are defined numerically in the following table. No other points exist for these functions other than the points given. Find the output values for each of the other combinations of functions indicated. Fill in as many points as are defined based on the given data. Use the same input values for all functions.

x	$f(x)$	$g(x)$	$(f+g)(x)$	$(f-g)(x)$	$f(x)\cdot g(x)$	$f^{-1}(x)$	$g(f(x))$	$f(g(x))$
0	0	−3						
1	2	−2						
2	4	−1						
3	6	0						
4	8	1						
5	10	2						
6	12	3						
7	14	4						
8	16	5						

4. Remember the race between the tortoise and the hare? Well, their friends and families have come to cheer them on, and have positioned themselves at various places along the course. Because rabbits are quick and eager to know the outcome of the race, more of them have congregated towards the end of the course. Because tortoises are slow and more anxious to cheer their champion off to a good start, more of them have congregated at the beginning of the race. In fact, the density (or amount of animals/meter) of tortoises and rabbits along the course, as a function of the distance from the starting line, is given by the following functions:

The tortoise: $\boldsymbol{a(d) = 243 \cdot \left(\frac{1}{3}\right)^{\frac{1}{20}d}}$ (a is in tortoises per meter, d in meters)

The hare: $\boldsymbol{a(d) = 2^{\frac{1}{10}d}}$ (a is in rabbits per meter, d in meters)

The distance from the starting line, as a function of the elapsed time since the start of the race, is given for the tortoise and the hare by the following functions.

The tortoise: $\boldsymbol{d(t) = 2^t}$ (d in meters, t in seconds)

The hare: $\boldsymbol{d(t) = t^2}$ (d in meters, t in seconds)

The tortoise and the hare are anxious to know how many of their friends and family they are passing at any instant in time along the race.

a. Create functions for the tortoise and for the hare that will calculate the number of tortoises or rabbits they will pass at any time, t, after the race begins. Include a reasonable domain for each function.

Tortoise:

Hare:

b. If the race is 100 meters long, create a function that will tell how many spectators, rabbits and tortoises, are watching at any distance away from the start of the race?

c. Who is passing more friends and family members 5 seconds after the race began: the tortoise or the hare?

Integrated Math 3
Module 7
Modeling with Functions
Ready, Set, Go! Homework

Adapted from

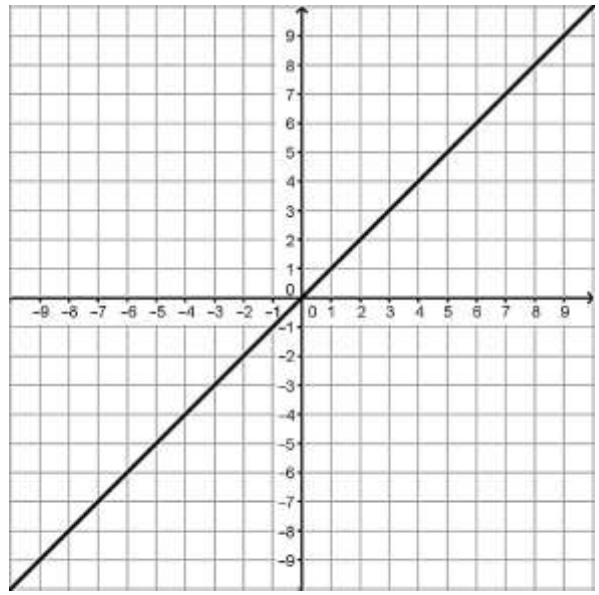
The Mathematics Vision Project:
Scott Hendrickson, Joleigh Honey, Barbara Kuehl,
Travis Lemon, Janet Sutorius

Ready, Set, Go!

Ready

Topic: Transformations

1. Graph the following linear functions on the grid. The equation $y = x$ has been graphed for you. For each new equation, explain what the number 2 does to the graph of $y = x$. Pay attention to the y-intercept, the x-intercept, and the slope. **Identify what changes in the graph and what stays the same.**

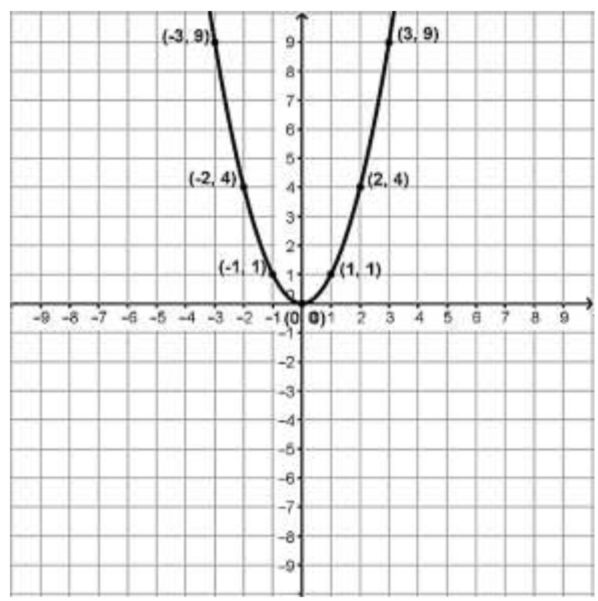


   a. $y_1 = x + 2$

   b. $y_2 = x - 2$

   c. $y_3 = 2x$

2. Graph the following quadratic functions on the grid. The equation $y = x^2$ has been graphed for you. For each new equation, explain what the number 3 does to the graph of $y = x^2$. Pay attention to the y-intercept, the x-intercept(s), and the rate of change. **Identify what changes in the graph and what stays the same.**



   a. $y_1 = x^2 + 3$

   b. $y_2 = x^2 - 3$

   c. $y_3 = (x - 3)^2$

   d. $y_4 = (x + 3)^2$

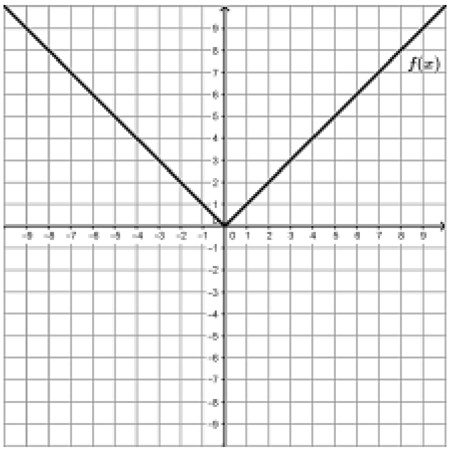
   e. $y_5 = 3x^2$

Set

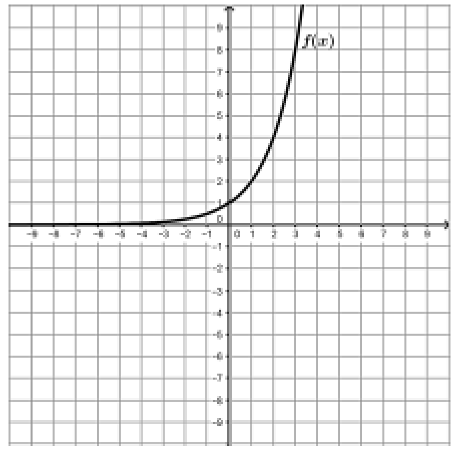
Topic: Transformations on parent functions.

The graph of the parent function is provided. Use this graph to sketch the transformed function on the same set of axes.

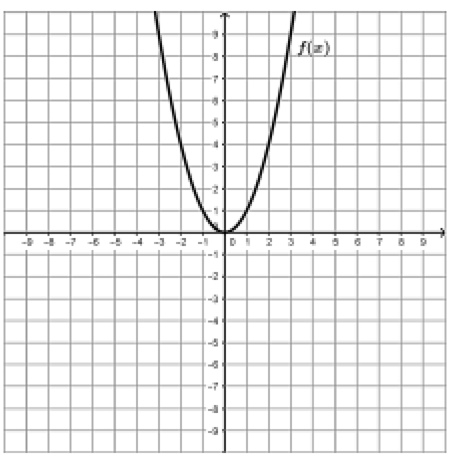
3. $f(x) = |x|$ and $g(x) = |x + 3|$



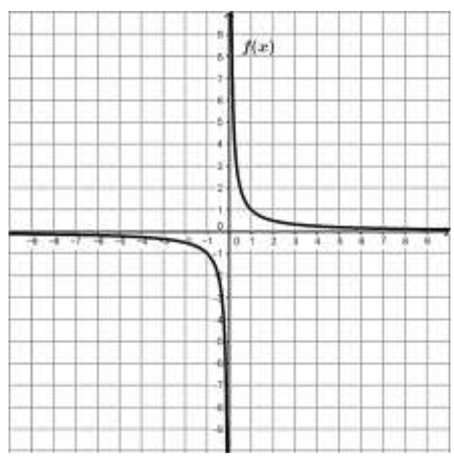
4. $f(x) = 2^x$ and $g(x) = 2^{-x}$



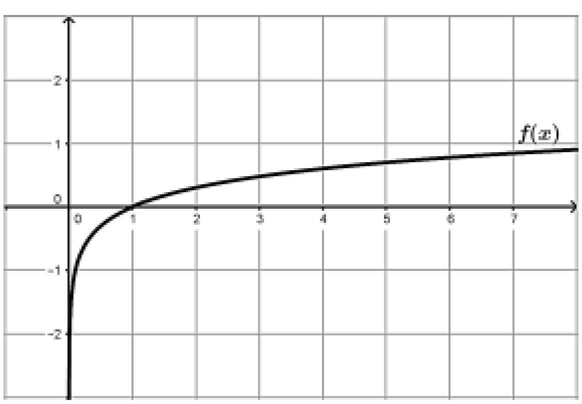
5. $f(x) = x^2$ and $g(x) = -\frac{1}{2}x^2 + 5$



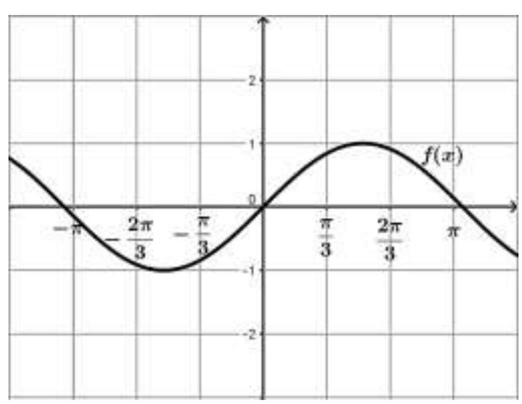
6. $f(x) = \frac{1}{x}$ and $g(x) = -\frac{1}{x}$



7. $f(x) = \log x$ and $g(x) = -1 + \log x$



8. $f(x) = \sin x$ and $g(x) = 2\sin\left(x + \frac{\pi}{2}\right)$



Go

Topic: Evaluating functions

Find the function values: $f(-2), f(0), f(1), f(3)$. Indicate if the function is undefined for a given value of *x*.

9. $f(x) = |x + 5|$

$f(-2) =$, $f(0) =$,

$f(1) =$, $f(3) =$

10. $f(x) = |x - 2|$

$f(-2) =$, $f(0) =$,

$f(1) =$, $f(3) =$

11. $f(x) = x|x|$

$f(-2) =$, $f(0) =$,

$f(1) =$, $f(3) =$

12. $f(x) = 3^x$

$f(-2) =$, $f(0) =$,

$f(1) =$, $f(3) =$

13. $f(x) = 3^{x+2}$

$f(-2) =$, $f(0) =$,

$f(1) =$, $f(3) =$

14. $f(x) = (3^x) + x$

$f(-2) =$, $f(0) =$,

$f(1) =$, $f(3) =$

15. $f(x) = \frac{x}{x}$

$f(-2) =$, $f(0) =$,

$f(1) =$, $f(3) =$

16. $f(x) = \frac{x}{(x-4)}$

$f(-2) =$, $f(0) =$,

$f(1) =$, $f(3) =$

17. $f(x) = \frac{x}{(x+2)} - 5$

$f(-2) =$, $f(0) =$,

$f(1) =$, $f(3) =$

18. $f(x) = \log_3 x$

$f(-2) =$, $f(0) =$,

$f(1) =$, $f(3) =$

19. $f(x) = \log_7 (7)^x$

$f(-2) =$, $f(0) =$,

$f(1) =$, $f(3) =$

20. $f(x) = x \log_{10} 1000$

$f(-2) =$, $f(0) =$,

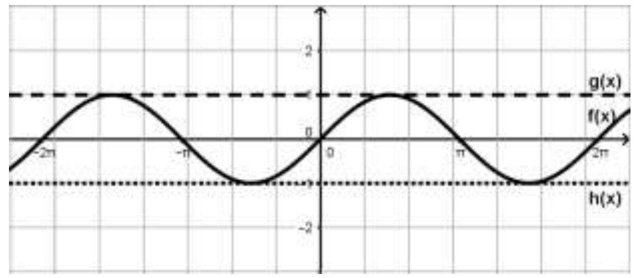
$f(1) =$, $f(3) =$

Ready, Set, Go!

Ready

Topic: Function boundaries.

1. The black solid curve in the graph at the right shows the graph of $f(x) = \sin x$.

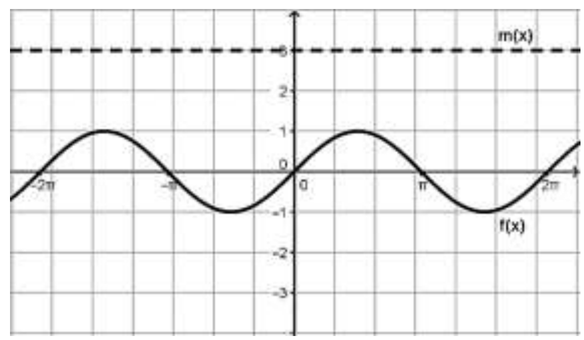


   Write the equation of the dashed line labeled $g(x)$.

   Write the equation of the dotted line labeled $h(x)$.

   List everything you notice about these three graphs.

2. The black solid curve in the graph at the right shows the graph of $f(x) = \sin x$.



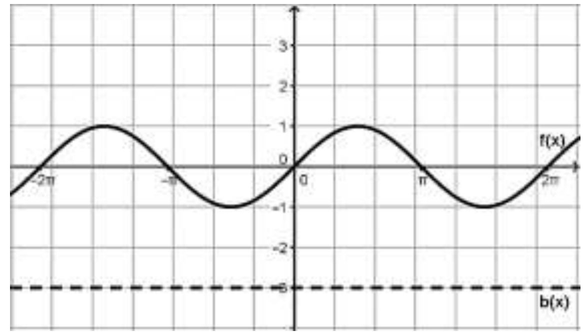
   Write the equation of the dashed line labeled $m(x)$.

   Sketch the graph of $f(x) \cdot m(x)$ on the same grid.

   What is the equation of $f(x) \cdot m(x)$?

   Would the line $y = -3$ also be a boundary line for your sketch of $f(x) \cdot m(x)$? Explain.

3. The black solid curve in the graph at the right shows the graph of $f(x) = \sin x$.



Write the equation of the dashed line labeled $b(x)$.

Sketch the graph of $f(x) \cdot b(x)$ on the same grid.

What is the equation of $f(x) \cdot b(x)$?

Would the line $y = 3$ also be a boundary line for your sketch of $f(x) \cdot b(x)$? Explain.

How does the graph of $f(x) \cdot b(x)$ differ from the graph of $f(x) \cdot m(x)$ from question 2?

4. In questions 2 and 3, how do the horizontal dashed lines, $b(x)$ and $m(x)$, affect the graph of $f(x)$?

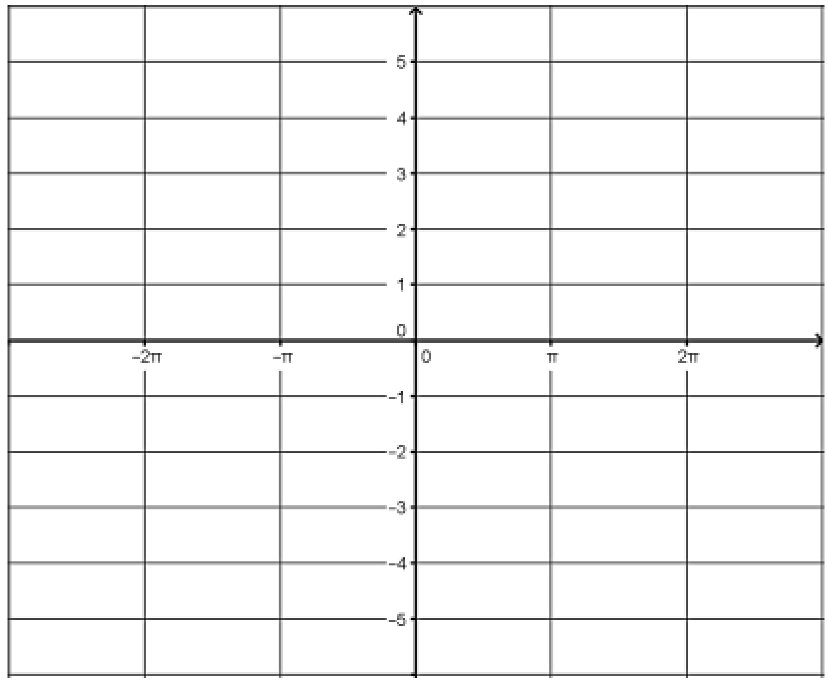
Set

Topic: Combining functions

5. $f(x) = x$ $\qquad g(x) = \sin x$ $\qquad h(x) = f(x) + g(x)$

Some values for $f(x)$ *and* $g(x)$ are given. Fill in the values for $h(x)$. Then graph $h(x) = x + \sin x$ with a smooth curve.

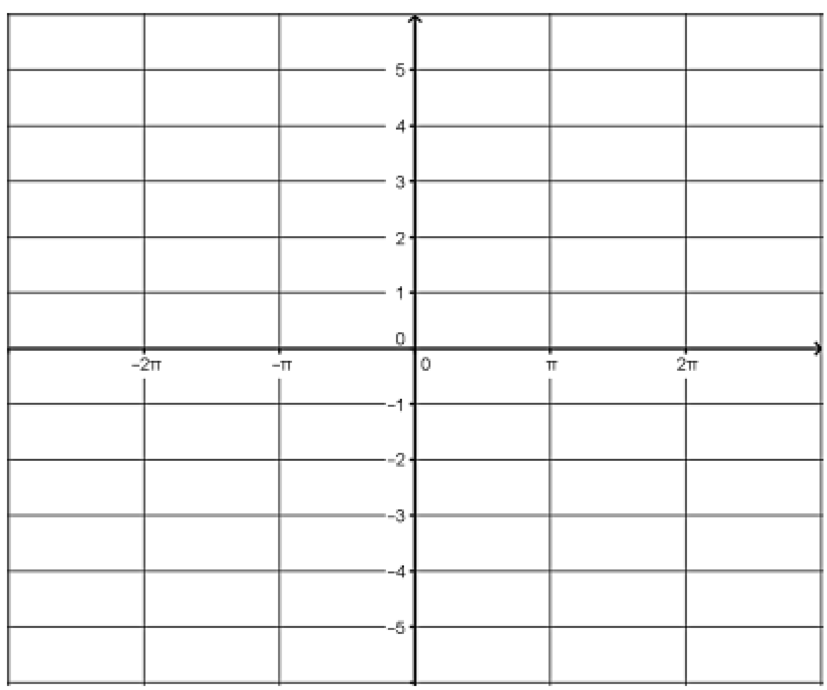
x	$f(x)$	$g(x)$	$h(x)$
-2π	-6.28	0	
$-\frac{3\pi}{2}$	-4.71	1	
$-\pi$	-3.14	0	
$-\frac{\pi}{2}$	-1.57	-1	
0	0	0	
$\frac{\pi}{2}$		1	
π		0	
$\frac{3\pi}{2}$		-1	
2π		0	



6. $f(x) = x$ $\qquad g(x) = \sin x$

Now graph $k(x) = f(x) \cdot g(x)$ or $k(x) = x \cdot \sin x$

x	$f(x)$	$g(x)$	$k(x)$
-2π	-6.28	0	
$-\frac{3\pi}{2}$	-4.71	1	
$-\pi$	-3.14	0	
$-\frac{\pi}{2}$	-1.57	-1	
0	0	0	
$\frac{\pi}{2}$		1	
π		0	
$\frac{3\pi}{2}$		-1	
2π		0	



Match the equations with the appropriate graph. Describe the features of the graph that helped you match the equations.

7. $f(x) = |x^2 - 4|$

Key features:

8. $g(x) = -x + 5\sin x$

Key features:

9. $h(x) = 4|\sin x|$

Key features:

10. $d(x) = (10 - x^2) + 5\sin x$

Key features:

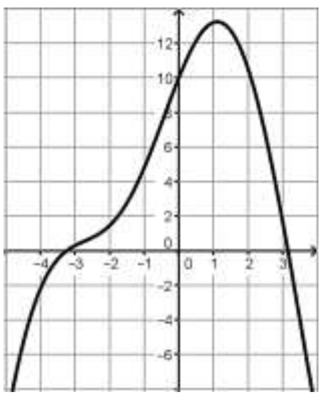
11. $w(x) = -x \cdot 2\sin x$

Key features:

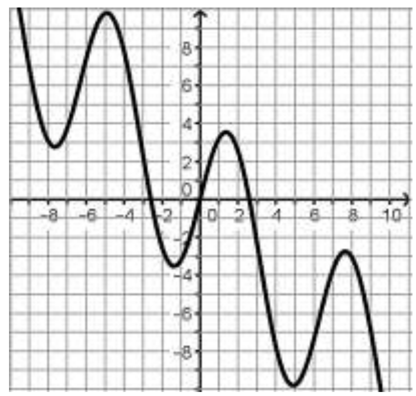
12. $r(x) = (2x - 4) + |x|$

Key features:

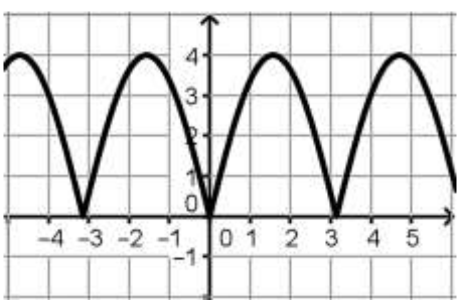
a.



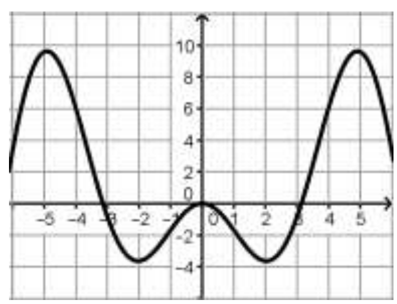
b.



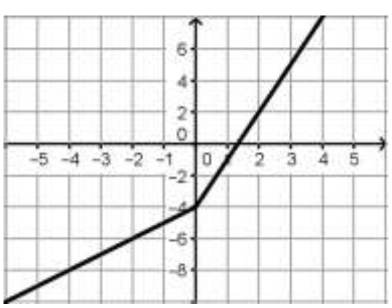
c.



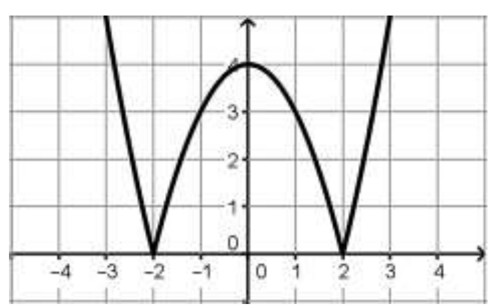
d.



e.



f.



Go

Topic: Families of functions

The chart below names five families of functions and the parent function. The parent is the equation in its simplest form. In the right hand column is a list of key features of the functions in random order. Match each key feature with the correct function. A key feature may relate to more than one function.

Family	*Parent(s)*	*Key Features*
13. Linear __________	$y = x$	a. The ends of the graph have the same behavior. b. The graphs may have a horizontal asymptote and a vertical asymptote. c. The graph only has a horizontal asymptote.
14. Quadratic __________	$y = x^2$	d. These functions either have both a local maximum and minimum or no local maximum and minimum. e. The graph is usually defined in terms of its slope and y-intercept. f. The graph has either a maximum or a minimum but not both.
15. Cubic __________	$y = x^3$	g. As x approaches $-\infty$, the function values approach the x-axis. h. The ends of the graph have opposite behavior. i. The rate of change of this graph is constant.
16. Exponential __________	$y = 2^x$ $y = 3^x$ Etc.	j. The rate of change of this graph is constantly changing. k. This graph has a linear rate of change. l. These functions are of degree 3.
17. Rational __________	$y = \frac{1}{x}$	m. The variable is an exponent. n. These functions contain fractions with a polynomial in both the numerator and denominator. p. The constant will always be the y-intercept.

Name **Modeling with Functions | 7.3**

Ready, Set, Go!

Ready

Topic: Evaluating functions

Evaluate each function. Simplify your answers when possible. State *undefined* when applicable.

1. $f(x) = x^2 - 8x$

   a. $f(0)$ b. $f(-10)$ c. $f(5)$ d. $f(8)$ e. $f(x+2)$

2. $g(x) = \frac{3x-5}{x}$

   a. $g(-1)$ b. $g(10)$ c. $g\left(\frac{1}{3}\right)$ d. $g(0)$ e. $g(2x+4)$

3. $h(x) = \sin x$

   a. $h(\pi)$ b. $h\left(\frac{3\pi}{2}\right)$ c. $h\left(\frac{11\pi}{6}\right)$ d. $h\left(\frac{5\pi}{4}\right)$ e. $h(\cos^{-1}\left(\frac{1}{2}\right), x < \pi)$

4. $w(x) = \tan x$

   a. $w(\pi)$ b. $w\left(\frac{3\pi}{2}\right)$ c. $w\left(\frac{7\pi}{6}\right)$ d. $w\left(\frac{3\pi}{4}\right)$ e. $w\left(\cos^{-1}\left(-\frac{1}{2}\right), x < \pi\right)$

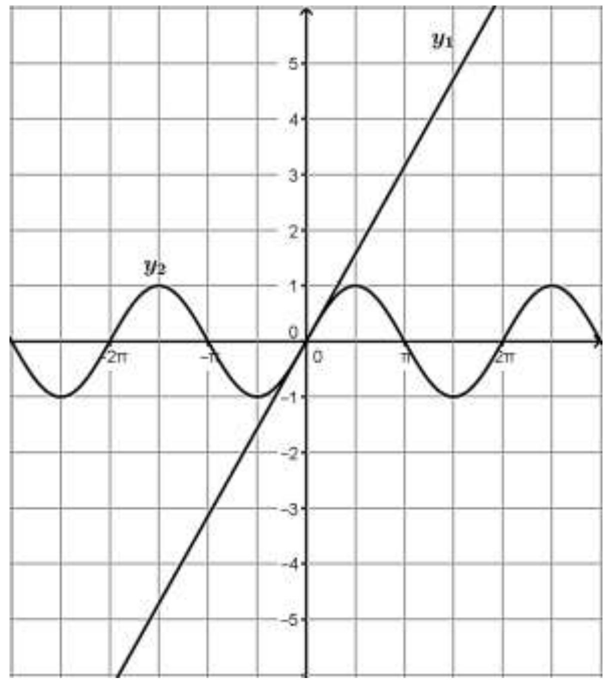
Set

Topic: Dampening functions

Two functions are graphed. Graph a third function by multiplying the two functions together. Use the table of values to assist you. It may help you to change the function values to decimals.

5.

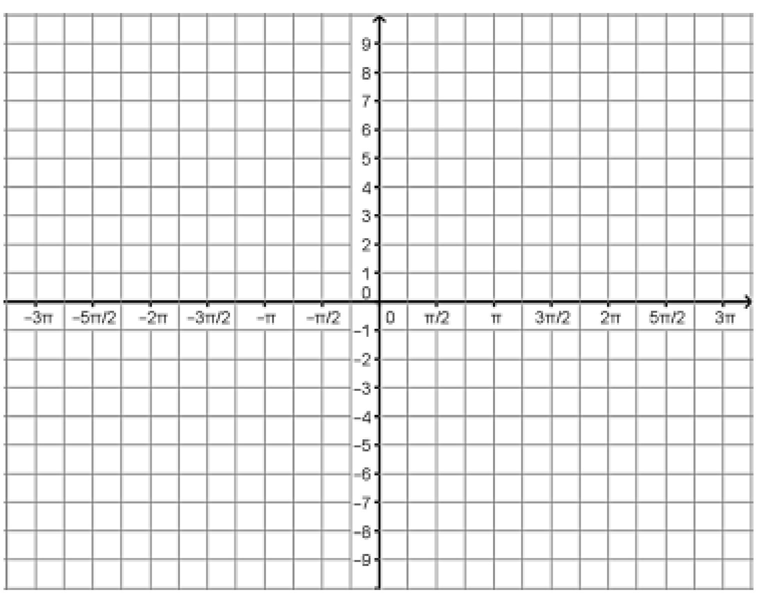
x	$y_1 = x$	$y_2 = \sin x$	$y_3 = x \cdot \sin x$
-2π			
$-\frac{3\pi}{2}$			
$-\pi$			
$-\frac{\pi}{2}$			
0			
$\frac{\pi}{2}$			
π			
$\frac{3\pi}{2}$			
2π			



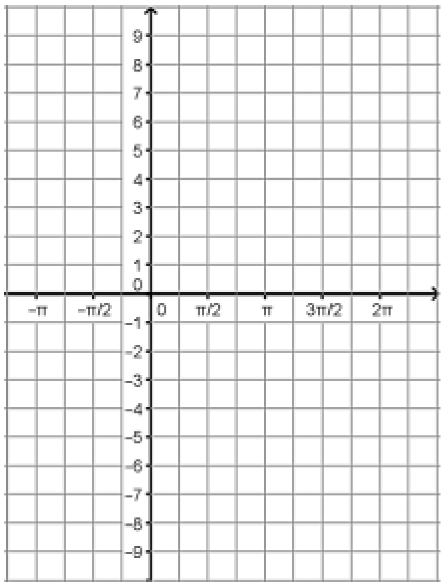
Topic: Graphing products and sums of functions.

For each set of functions below, graph the indicated operation. Be sure to use the features of each individual function to help you complete the graph.

6. $f(x) = 2x - 8$ and $g(x) = 3\sin(2x)$. Graph $f(x) + g(x)$.



7. $f(x) = 8 \cdot \left(\frac{1}{2}\right)^x$ and $g(x) = \cos(4x)$. Graph $f(x) \cdot g(x)$.



Go

Topic: Measures of central tendency (Mean, median, mode)

During salary negotiations for teacher pay in a rural community, the local newspaper headlines announced: **Greedy Teachers Demand More Pay!** The article went on to report that teachers were asking for a pay hike even though district employees, including teachers, were paid an average of $70,000 per year, while the average annual income for the community was calculated to be $55,000 per household. The 65 school teachers in the district responded by declaring that the newspaper was spreading false information.

Use the table below to explore the validity of the newspaper report.

Job Description	Number Having Job	Annual Salary
Superintendent	1	$258,000
Business Administrator	1	$250,000
Financial Officer	1	$205,000
Transportation Coordinator	1	$185,000
District secretaries	5	$ 55,000
School Principals	5	$200,000
Assistant Principals	5	$175,000
Guidance Counselors	10	$ 85,000
School Nurse	5	$ 83,000
School Secretaries	10	$ 45,000
Teachers	65	$ 48,000
Custodians	10	$ 40,000

8. Which measure of central tendency (mean, median, mode) do you think the newspaper used to report the teachers' salaries? Justify your answer. Note: not all measures of central tendency need to be calculated in order to answer this question. You might create a histogram to help you analyze the data.

9. Which measure of central tendency do you think the teachers would use to support their argument? Justify your answer.

10. Which measure gives the clearest picture of the salary structure in the district? Justify.

11. Make up a headline for the newspaper that would be more accurate.

Name **Modeling with Functions | 7.4**

Ready, Set, Go!

Ready

Topic: Recognizing operations on a variable

Each function below contains 2 functions. One of the functions will be "inside" the other. Identify the "inner" function as *u* by writing *u* = _________. Then substitute *u* into the outer function so that the new function is of the form $h(u)$.

Example:

Given: $h(x) = 5x^3$

I can see two functions acting on *x*. I will let $u = x^3$. Then $h(u) = 5u$.

1. Would the answer in the example have been different if you were given $(5x)^3$? Explain

2. $h(x) = (x - 6)^2$

   $u =$

   $h(u) =$

3. $h(x) = \tan(x + 4)$

   $u =$

   $h(u) =$

4. $h(x) = \sqrt[3]{(2x - 7)}$

   $u =$

   $h(u) =$

5. $h(x) = -9(x + 5)$

   $u =$

   $h(u) =$

6. $h(x) = \frac{5}{x^2}$

   $u =$

   $h(u) =$

7. $h(x) = (\sin x)^4$

   $u =$

   $h(u) =$

Set

Topic: Creating formulas for composite functions

Note: $\boldsymbol{f(g(x)) = (f \circ g)(x)}$

8. Let $f(x) = 2x^2 - 4$ and $g(x) = 5x$. Find:

   a. $(f \circ g)(x)$ b. $(g \circ f)(1)$ c. $f(f(-2))$ d. $g(g(-1))$

9. Let $f(x) = \frac{8}{x-3}$ and $g(x) = \frac{15}{x+1}$ Find:

   a. $f(g(x))$ b. $g(f(x))$ c. $(f \circ f)(x)$ d. $(g \circ g)(x)$

10. Use the functions in question 9 to find:

    a. $(f \circ g)(2)$ b. $(g \circ f)(-5)$

11. Use the functions in question 9. Describe the domains for:

    a. $(f \circ g)(x)$ b. $(g \circ f)(x)$ c. $(f \circ f)(x)$ d. $(g \circ g)(x)$

12. What makes the domain for each composition different?

Go

Topic: Writing equations of polynomials given the degree and the roots

Write the equation of the polynomial with the given features. Hint: Use the roots and leading coefficient to first write the function in factored form $f(x) = a(x - p_1)(x - p_2)(x - p_3) \ldots$

	Degree of Polynomial	Given Roots (you may have to determine others):	Leading Coefficient	Equation in Standard Form:
13.	3	$-2, 1,$ and -1	3	
14.	4	$(2 + i), 4, 0$	1	
15.	5	1 multiplicity 2, -1 multiplicity 2, and 3	-1	
16.	4	$(3 - i), \sqrt{2}$	-2	

Ready, Set, Go!

Ready

Topic: Using a table to find the value of a composite function

Use the table to find the indicated function values.

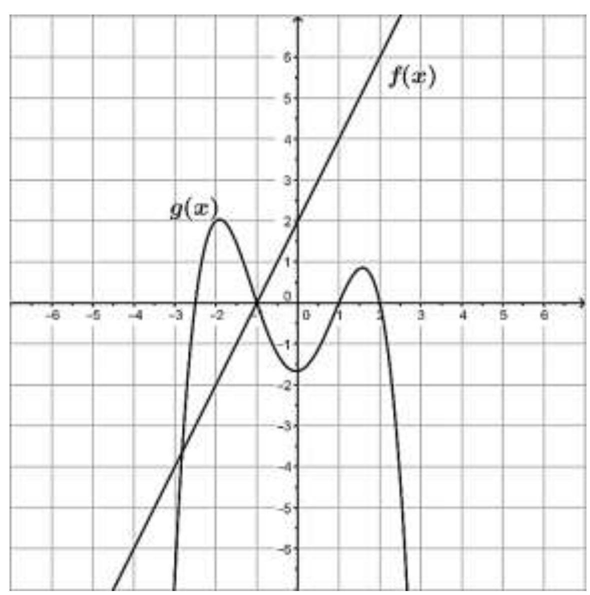
x	$f(x)$	$g(x)$
-2	2	3
-1	1	-2
0	3	-24
1	-1	-1
2	0	-8
3	19	0

Example: Find $g(f(1))$.
$f(1) = -1$
$g(-1) = -2$
Therefore, $g\big(f(1)\big) = -2$

1. $f(g(3))$
2. $f(g(1))$
3. $g(f(-2))$
4. $g(f(-1))$
5. $g(f(0))$
6. $g(g(-2))$
7. $f(f(0))$
8. Do the graphs of $f(x)$ and $g(x)$ ever intersect each other? How do you know?

Use the graph at the right to find the indicated values.

9. $f(g(-2))$
10. $f(g(-1))$
11. $f(g(0.5))$
12. $f(f(0))$



Set

Topic: Creating a composite function given its components

Let $f(x) = x^2, g(x) = 5x$, and $h(x) = \sqrt{x} + 2$. Express each function as a composite of *f, g* and/or *h.*

13. $A(x) = x^4$

14. $C(x) = 5x^2$

15. $P(x) = x + 2$

16. $R(x) = 5\sqrt{x} + 10$

17. $Q(x) = 25x$

18. $M(x) = 25x^2$

19. $D(x) = \sqrt{\sqrt{x} + 2} + 2$

20. $B(x) = x + 4\sqrt{x} + 4$

21. $K(x) = \sqrt{5x} + 2$

22. Find the each composition using the following functions:

$h(x) = 2x - 3,\ m(x) = -x^2 + 4x,\ p(x) = \sqrt{2x + 5}$

a. $m\big(h(6)\big)$

b. $p\Big(h\big(m(2)\big)\Big)$

c. $h\Big(p\big(m(x)\big)\Big)$

d. $m\Big(p\big(h(x)\big)\Big)$

Go

Topic: Finding the zeros of polynomial and rational equations

Solve for all of the values of *x*. Identify any restrictions on *x*.

23. $x^2 + 6 = 5x$

24. $5x^3 = 45x$

25. $x^4 - 26x^2 + 25 = 0$

26. $1 + \frac{1}{x} = \frac{12}{x^2}$

27. $\frac{x}{6} - \frac{1}{2} - \frac{3}{x} = 0$

28. $\frac{1}{x^2} = 9$

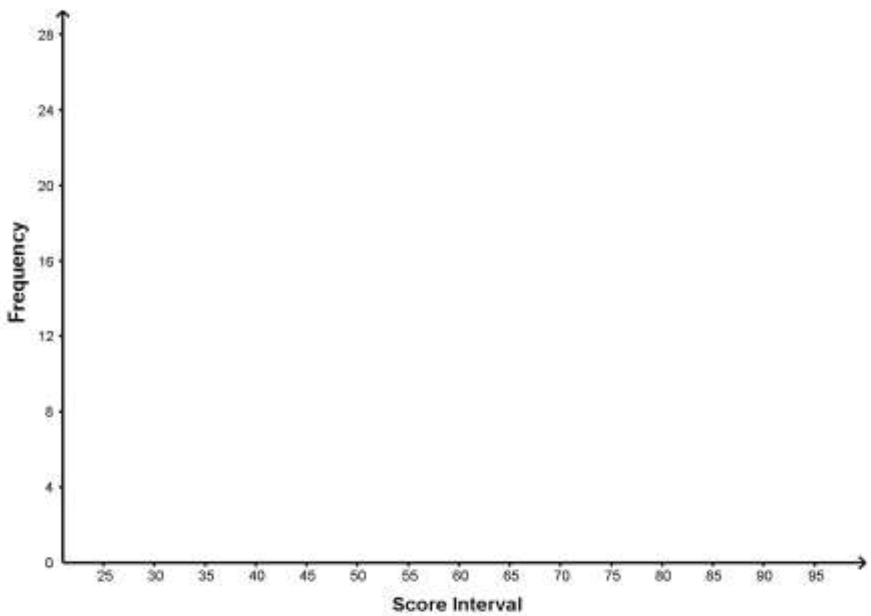
Ready, Set, Go!

Ready

Topic: Histograms

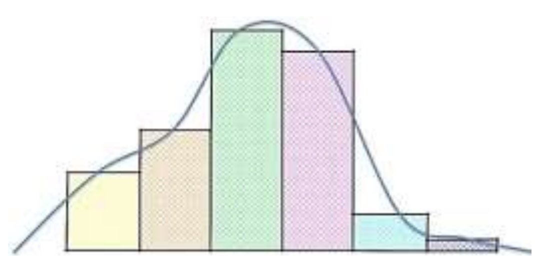
One hundred forty-four college freshmen were given a math placement exam with 100 possible points. The results show that 56 different scores were made, ranging from 24 to 96. The scores were grouped in intervals as shown in the following table:

1. Make a histogram of the grouped data in the chart. Note: The midpoint of each cell is given in the horizontal axis. The sides of the cells will match the score interval. Frequency is the vertical height.



Score Interval	Midpoint of Interval	Frequency of Interval
92.5-97.5	95	2
87.5-92.5	90	4
82.5-87.5	85	10
77.5-82.5	80	13
72.5-77.5	75	21
67.5-72.5	70	26
62.5-67.5	65	18
57.5-62.5	60	15
52.5-57.5	55	12
47.5-52.5	50	8
42.5-47.5	45	3
37.5-42.5	40	3
32.5-37.5	35	4
27.5-32.5	30	4
22.5-27.5	25	1

2. Locate the midpoint at the top of each cell and connect each consecutive midpoint with straight line segments. The resulting figure is called a ***frequency polygon***. If you smooth the line segments out into a smooth curve, you will create a ***frequency curve.*** Make a frequency curve on your histogram. It should look something like the figure on the right.



Set

Topic: Identifying the 2 functions that make up a composite function

Find functions *f* and *g* so that $f \circ g = H$

3. $H(x) = \sqrt{x^2 + 5x - 4}$

4. $H(x) = \left(3 - \frac{1}{x}\right)^2$

5. $H(x) = (3x - 7)^4$

6. $H(x) = |5x^2 - 78|$

7. $H(x) = \frac{2}{3 - x^5}$

8. $H(x) = (\tan x)^2$

9. $H(x) = \tan(x^2)$

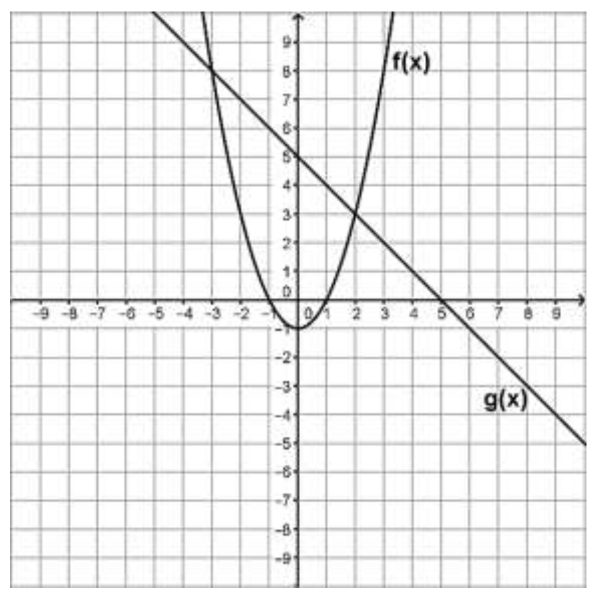
10. $H(x) = \sqrt{\frac{1}{6x}}$

11. $H(x) = 9(4x - 8) + 1$

Go

Topic: Finding function values given the graph

Use the graph to find all of the missing values.



12. $f(\quad) = 8$

13. $g(\quad) = 5$

14. $f(\quad) = -1$

15. $g(\quad) = 0$

16. $f(-1) =$

17. $g(0) =$

18. $f(x) = g(x)$

19. $f(x) - g(x) = 0$

20. $f(x) \cdot g(x) = 0$

21. $f(2) + g(2) =$

22. $f(0) - g(0) =$

Integrated Math 3
Module 8
Statistics

Adapted from

The Mathematics Vision Project:
Scott Hendrickson, Joleigh Honey, Barbara Kuehl,
Travis Lemon, Janet Sutorius

Module 8 Overview

Prerequisite Concepts and Skills:

- Describe and compare data distributions/sets in a context, table, and/or graph
- Represent and interpret data using a histogram, box plot, and scatterplot
- Analyze a set of data using mean, median, and mode
- Two-way tables
- Computing probabilities – basic, conditional, independent/dependent, and union
- Describing center, spread, and shape of distributions
- Creating and interpreting histograms

Summary of the Concepts & Skills in Module 8:

- Identifying features of normal distributions
- Interpreting contexts using normal distributions
- 68-95-99.7 Empirical Rule for normal distributions
- Comparing normal distributions using z-scores
- Ranking data based on means, standard deviations, and z-scores
- Identifying methods of sampling
- Determining which methods of sampling should be used in different contexts
- Using simulations to determine the likelihood of outcomes

Content Standards and Standards for Mathematical Practice Covered:

- Content Standards: S.ID.4, S.IC.1, S.IC.2, S.IC.3
- Standards for Mathematical Practice:
  1. Make sense of problems and persevere in solving them.
  2. Reason abstractly and quantitatively.
  3. Construct viable arguments and critique the reasoning of others.
  4. Model with mathematics.
  5. Use appropriate tools strategically.
  6. Attend to precision.
  7. Look for and make use of structure.
  8. Look for and express regularity in repeated reasoning.

Module 8 Vocabulary:

- Point of inflection
- Distribution curve
- Normal distribution
- Standard deviation
- Mean
- Median
- Inference
- Systematic sample
- Stratified random sample
- Volunteer sample
- Observational study
- Control group
- Simulation
- Frequency distribution
- Law of large numbers
- Mode
- z-score
- Standard normal distribution
- Population
- Random sample
- Parameter of interest
- Simple random sample
- Cluster sample
- Convenience sample
- Survey
- Experiment
- Experimental group
- Symmetric
- 68-95-99.7 Empirical rule

Module 8 – Statistics

8.1 *Understand normal distributions and identify their features* **(S.ID.4)**
Warm Up: Measures of Central Tendency and Standard Deviation
Classroom Task: What is Normal? – A Develop Understanding Task
Ready, Set, Go Homework: Statistics 8.1

8.2 *Use the features of a normal distribution to make decisions* **(S.ID.4)**
Warm Up: Skewed vs. Normally Distributed Data
Classroom Task: Just ACT Normal – A Solidify Understanding Task
Ready, Set, Go Homework: Statistics 8.2

8.3 *Compare normal distributions using z scores* and *Compare normal distributions using z scores and understanding of mean and standard deviation* **(S.ID.4)**
Warm Up: Y B Normal? – A Solidify Understanding Task
Classroom Task: Whoa! That's Weird! – A Practice Understanding Task
Ready, Set, Go Homework: Statistics 8.3

8.4 *Understand and identify different methods of sampling* **(S.IC.1, S.IC.3)**
Warm Up: Comparing Samples
Classroom Task: Would You Like to Try a Sample? – A Develop Understanding Task
Ready, Set, Go Homework: Statistics 8.4

8.5 *Explore different sampling methods in context* **(S.IC.3)**
Classroom Task: Rolling Down the River – A Solidify Understanding Task
Ready, Set, Go Homework: Statistics 8.5

8.6 *Identify the difference between survey, observational studies, and experiments* and *Use simulation to estimate the likelihood of an event* **(S.IC.2, S.IC.3)**
Warm up: Let's Investigate – A Solidify Understanding Task
Classroom Task: Slacker's Simulation – A Solidify Understanding Task
Set, Go Homework: Statistics 8.6

8.1 Resource Pages
Some Statistics for Reference

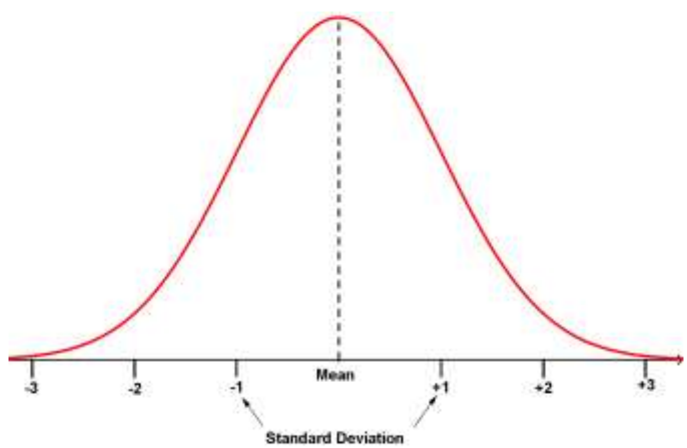
A lot of information can be obtained from looking at data plots and their distributions. It is important when describing data that we use context to communicate the **shape, center,** and **spread.**

Shape and spread:
- ***Modes***: uniform (evenly spread-no obvious mode), unimodal (one main peak), bimodal (two main peaks), or multimodal (multiple locations where the data is relatively higher than others).
- ***Skewed distribution***: when most data is to one side leaving the other with a 'tail'. Data is skewed to side of tail. If tail is on left side of data, then it is skewed left.
- ***Outliers***: values that stand away from a body of distribution. Values less than $Q1 - 1.5(Q3 - Q1)$ or greater than $Q3 + 1.5(Q3 - Q1)$ are considered outliers.
- ***Normal distribution***: curve is unimodal and symmetric.
- ***Variability***: values that are close together have low variability; values that are spread apart have high variability.

Center:
- Analyze the data and see if one value can be used to describe the data set. Normal distributions make this easy. If the data is not normally distributed, determine if there is a 'center' value that best describes the data. ***Bimodal*** or ***multimodal*** data may not have a center that would provide useful data.

The **standard deviation (σ)** of a set of data measures how "spread out" the data set is. In other words, it tells you whether all the data items bunch around close to the mean (μ) or if they are "all over the place." Standard deviation provides a good rule of thumb for deciding whether something is "rare."

Whenever a set of data is **normally distributed**, the mean and standard deviation have the following appearance:

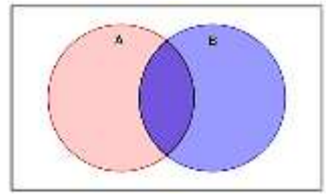


The point on the normal curve where the curve changes concavity is called the **point of inflection**. This point is located one standard deviation away from the mean.

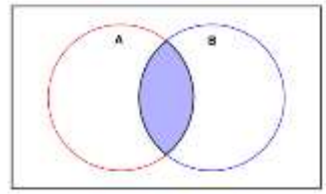
To calculate standard deviation:
1. Find the mean, $\boldsymbol{\mu}$
2. Find the difference between each data item and the mean, $\boldsymbol{x - \mu}$
3. Square each of the differences, $(\boldsymbol{x - \mu})^2$
4. Find the average (mean) of these squared differences.
5. Take the square root of this average.

Probability Vocabulary:

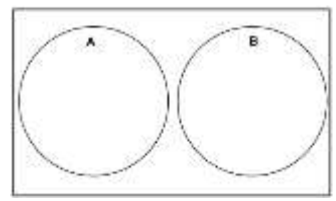
Union of A and B: A ∪ B is the set of all elements in either A **or** B



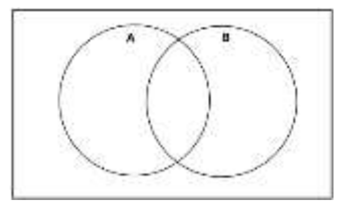
Intersection of A and B: A∩B is the set of all elements in both A **and** B



P(A∪B) = The probability that either A or B (or both) occur.
= P(A) + P(B) – P(A∩B) If A and B are **mutually exclusive**: P(A∪B) = P(A) + P(B)



A and B are **disjoint** or **mutually exclusive** if there is no intersection: P(A∩B) = 0



A and B are **joint** if the events are overlapping (one or more outcomes in common).

Complement of event A is all outcomes not contained in event A and is denoted by $\bar{A}$. It can be found using the formula: P($\bar{A}$)= 1 – P(A)

Conditional probability: The probability of event B occurring given that event A has already occurred. This can be read as "the probability of B given A" and is denoted by P(B|A).

P(B|A) $= \frac{P(A \cap B)}{P(A)}$

P(A∩B) represents the probability that **both** A **and** B occur. It follows from above that, P(A∩B) = P(A) · P(B|A)

A and B are **independent** if the occurrence of event A has no impact on the likelihood of event B. Therefore, P(B|A) = P(B) or P(A|B) = P(A)

It follows from above, that if A and B are **independent**, then P(A∩B) = P(A) · P(B)

- *Two events, A and B, are independent if* $P(A \text{ and } B) = P(A) \cdot P(B)$
- *Additionally, two events, A and B, are independent if* $P(A|B) = \frac{P(A \text{ and } B)}{P(B)} = P(A)$

8.1 Warm Up
Measures of Central Tendency & Standard Deviation

1. You and your friend have a friendly competition going on about the scores on your math quizzes. Both of your scores for the first five quizzes are given below.

   Your quiz scores: 18, 16, 19, 15, 17
   Friend's quiz scores: 20, 20, 13, 12, 17

   a. Find the mean of both sets of data.

      Your Mean:

      Friend's Mean:

   b. Find the median of both sets of data.

      Your Median:

      Friend's Median:

   c. Find the mode of both sets of data.

      Your Mode:

      Friend's Mode:

2. Find the standard deviation of the following set of data. Explain what this standard deviation tells us about the data.

   74, 72, 83, 96, 64, 79, 88, 69

3. The data set below gives the number of homeruns for the 10 batters who hit the most homeruns during the 2005 Major League Baseball regular season.

   51, 48, 47, 46, 45, 43, 41, 40, 40, 39

   a. Find the mean of the data set.

   b. Find the median of the data set.

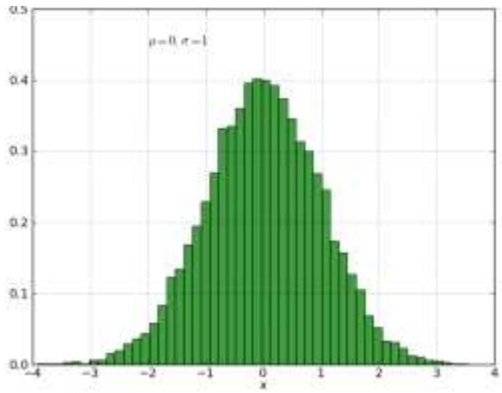
   c. Find the mode of the data set.

   d. Find the standard deviation of the data set.

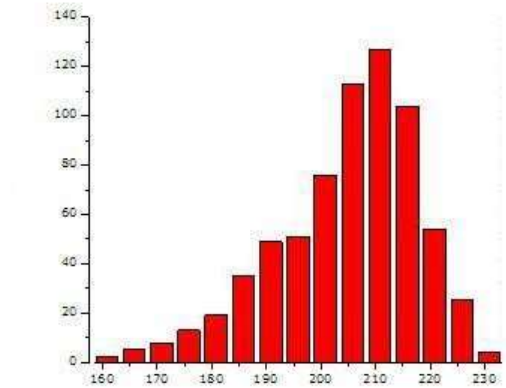
8.1 What is Normal?
A Develop Understanding Task

One very important type of data distribution is called a **"normal distribution."** In this task, you will be given pair of data distributions represented with histograms and distribution curves. In each pair, one distribution is normal and the other is not. Your job is to compare each of the distributions given and come up with a list of features for normal distributions.

1. This is normal:

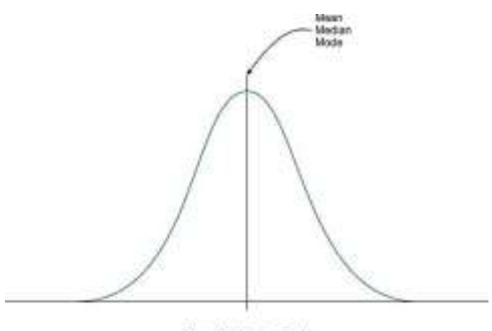


This is not:

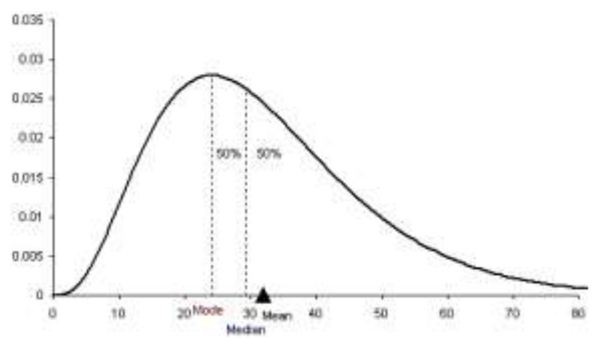


What differences do you see between these distributions?

2. This is normal:

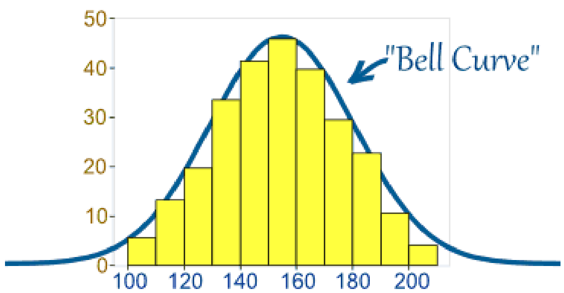


This is not:

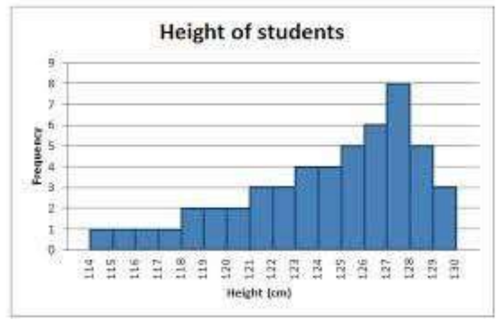


What differences do you see between these distributions?

3. This is normal:

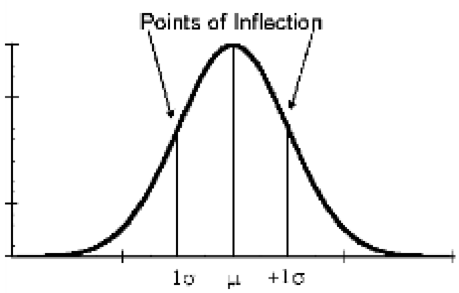


This is not:

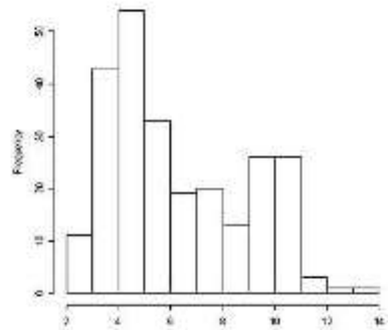


What differences do you see between these distributions?

4. This is normal:

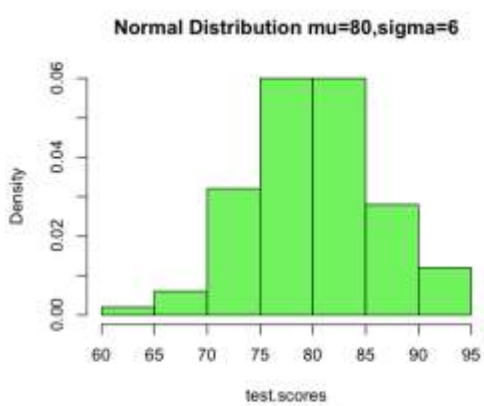


This is not:

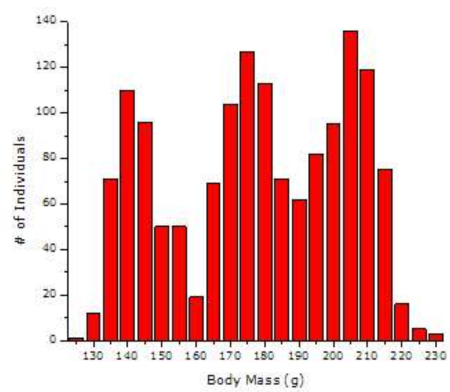


What differences do you see between these distributions?

5. This is normal:

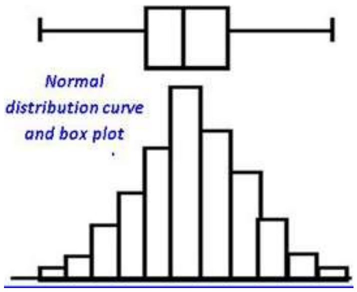


This is not:

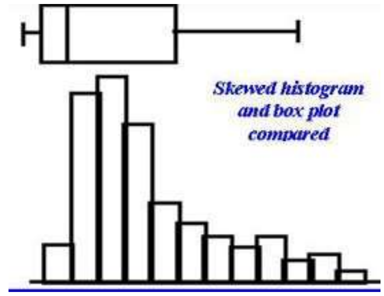


What differences do you see between these distributions?

6. This is normal:

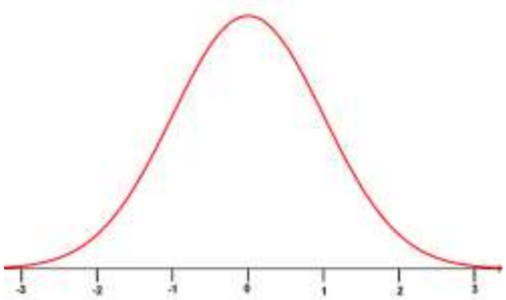


This is not:

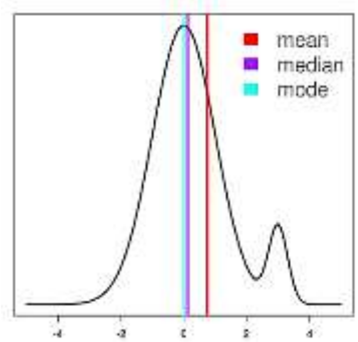


What differences do you see between these distributions?

7. This is normal:



This is not:



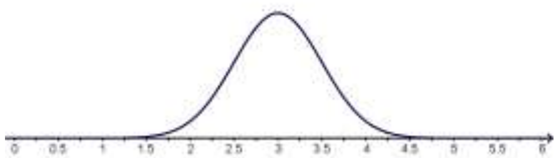
What differences do you see between these distributions?

8. Based upon the examples you have seen in #1-7, what are the features of a normal distribution?

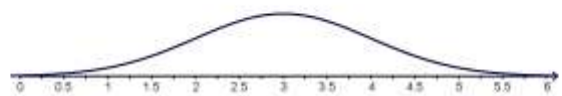
9. a. What does the **standard deviation** tell us about a distribution?

Each of the distributions shown below are normal distributions with the same mean but a different standard deviation.

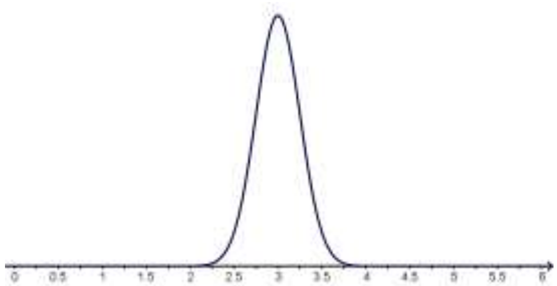
Mean = 3, Standard Deviation = 0.5



Mean = 3, Standard Deviation = 1



Mean = 3, Standard Deviation = 0.25

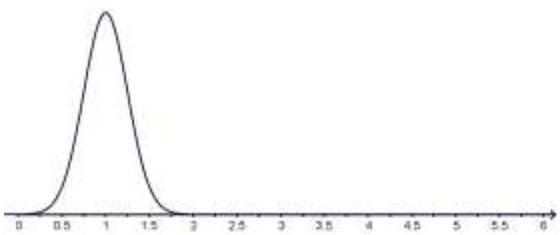


b. How does changing the **standard deviation** affect a normal curve? Why does it have this effect?

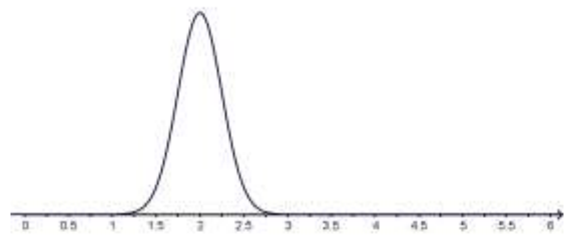
10. a. What does the **mean** tell us about a normal distribution?

Each of the distributions shown below are normal distributions with the same standard deviation but a different mean.

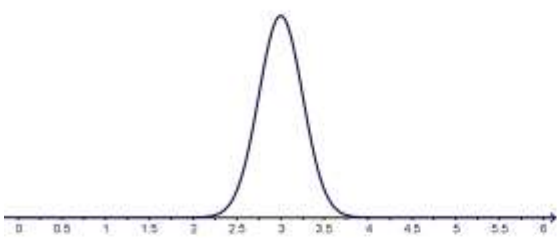
Mean = 1, Standard Deviation = 0.25



Mean = 2, Standard Deviation = 0.25



Mean = 3, Standard Deviation = 0.25



b. How does changing the mean affect a normal curve? Why does it have this effect?

11. Now that you have figured out some of the features of a normal distribution, determine if the following statements are true or false. In each case, explain your answer.

    a. The graph of a normal distribution depends on the mean and the standard deviation.

       True/False Explain:

    b. The mean, median, and mode are equal in a normal distribution.

       True/False Explain:

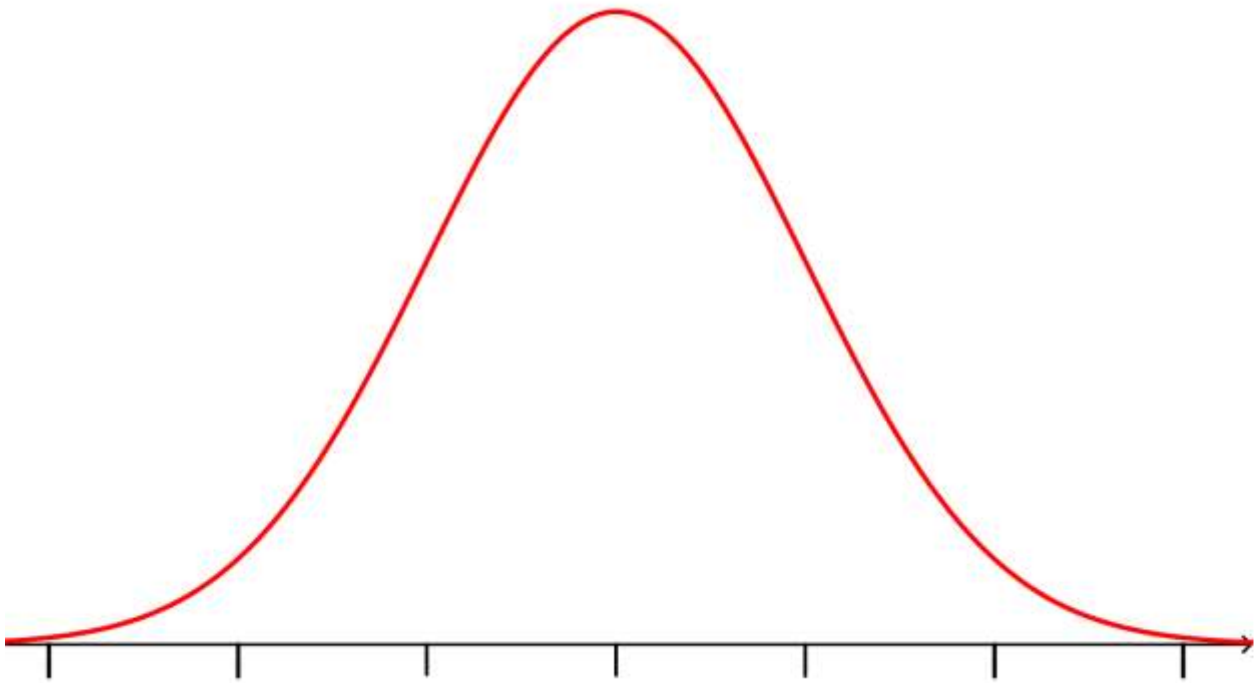
    c. A normal distribution is bimodal.

       True/False Explain:

Data on housefly wing lengths provide an excellent example of normally distributed data from the field of biometry. The data below was collected from a study conducted in 1955 and contains 100 sample wing lengths. For this data, the mean wing length is 4.55 mm and the standard deviation is 0.392 mm.

3.6, 3.7, 3.8, 3.8, 3.9, 3.9, 4.0, 4.0, 4.0, 4.0, 4.1, 4.1, 4.1, 4.1, 4.1, 4.1, 4.2, 4.2, 4.2, 4.2, 4.2, 4.2, 4.2, 4.3, 4.3, 4.3, 4.3, 4.3, 4.3, 4.3, 4.3, 4.4, 4.4, 4.4, 4.4, 4.4, 4.4, 4.4, 4.4, 4.4, 4.5, 4.5, 4.5, 4.5, 4.5, 4.5, 4.5, 4.5, 4.5, 4.5, 4.6, 4.6, 4.6, 4.6, 4.6, 4.6, 4.6, 4.6, 4.6, 4.6, 4.7, 4.7, 4.7, 4.7, 4.7, 4.7, 4.7, 4.7, 4.7, 4.8, 4.8, 4.8, 4.8, 4.8, 4.8, 4.8, 4.8, 4.9, 4.9, 4.9, 4.9, 4.9, 4.9, 4.9, 5.0, 5.0, 5.0, 5.0, 5.0, 5.0, 5.1, 5.1, 5.1, 5.1, 5.2, 5.2, 5.3, 5.3, 5.4, 5.5

12. Label the normal curve below with the mean and the values that are within 3 standard deviations left and right of the mean.



13. Calculate the percent of data that lies within one standard deviation of the mean.

14. Calculate the percent of data that lies within two standard deviations of the mean.

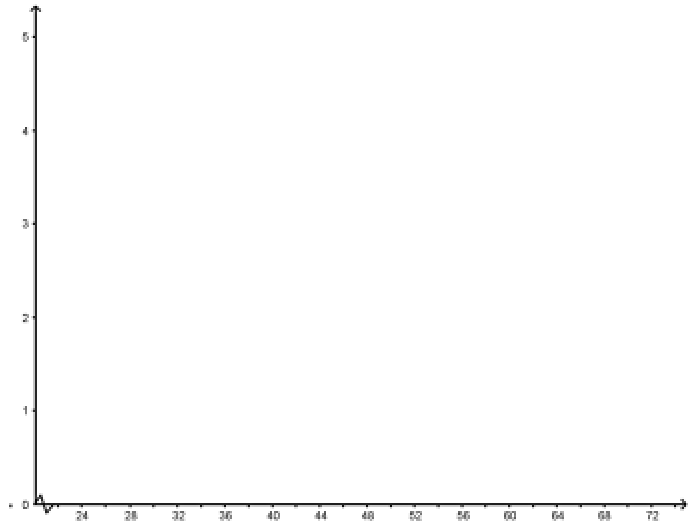
15. Calculate the percent of data that lies within three standard deviations of the mean.

8.2 Warm Up
Skewed vs Normally Distributed Data

Create a frequency histogram to determine if each data set is skewed or normally distributed (reminder: the number on the left of each interval is included, whereas the number on the right is not). Identify the mean of each data set and draw a vertical line on your histogram to visually represent the mean.

1. Data Set:
   25, 28, 30, 32, 34, 38, 41, 42, 45, 48, 48, 48, 50, 52, 52, 53, 54, 55, 56, 60, 60, 60, 60, 62, 62, 62, 62, 63

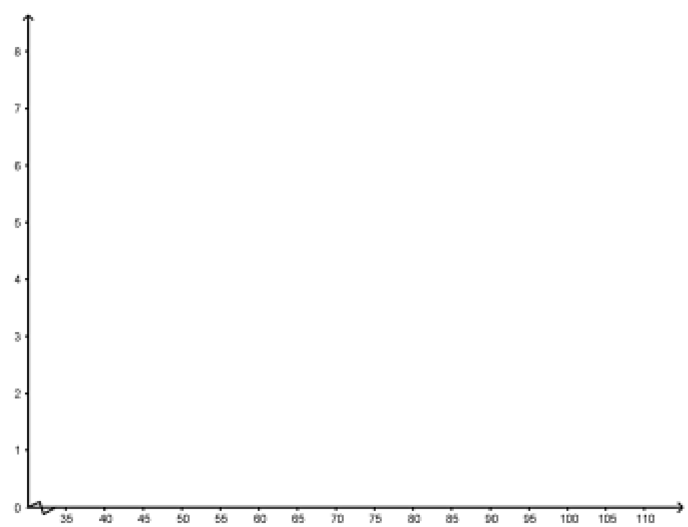
Interval	Frequency
22 -< 26	
26 -< 30	
30 -< 34	
34 -< 38	
38 -< 42	
42 -< 46	
46 -< 50	
50 -< 54	
54 -< 58	
58 -< 62	
62 -< 66	



Mean:

2. Data Set:
   37, 42, 44, 46, 47, 48, 50, 52, 53, 54, 55, 56, 58, 59, 60, 60, 61, 62, 62, 63, 64, 64, 65, 66, 66, 67, 67, 68, 68, 69, 70, 71, 72, 72, 73, 74, 75, 76, 78, 79, 80, 80, 83, 85, 86, 88, 90, 92

Interval	Frequency
35 -< 40	
40 -< 45	
45 -< 50	
50 -< 55	
55 -< 60	
60 -< 65	
65 -< 70	
70 -< 75	
75 -< 80	
80 -< 85	
85 -< 90	
90 -< 95	



Mean:

8.2 Just ACT Normal
A Solidify Understanding Task

1. One of the most common examples of a normal distribution is the distribution of scores on standardized tests like the ACT. In 2010, the mean score was 21 and the standard deviation was 5.2 (Source: National Center for Education Statistics).

   Use this information to sketch a normal distribution curve for this test.

2. Use technology to check your graph. Did you get the points of inflection in the right places? (Make adjustments on your graph, if necessary.)
   http://homepage.divms.uiowa.edu/~mbognar/applets/normal.html

3. In the task *What Is Normal*, you learned about the 68 – 95 – 99.7 rule. Use the rule to answer the following questions:

   a. What percentage of students scored below 21?

   b. About what percentage of students scored below 16?

   c. About what percentage of students scored between 11 and 26?

4. Your friend, Calvin, would like to go to a very selective college that only admits the top 1% of all student applicants. Calvin has good grades and scored 33 on the test. Do you think that Calvin's ACT score gives him a good chance of being admitted? Explain your answer.

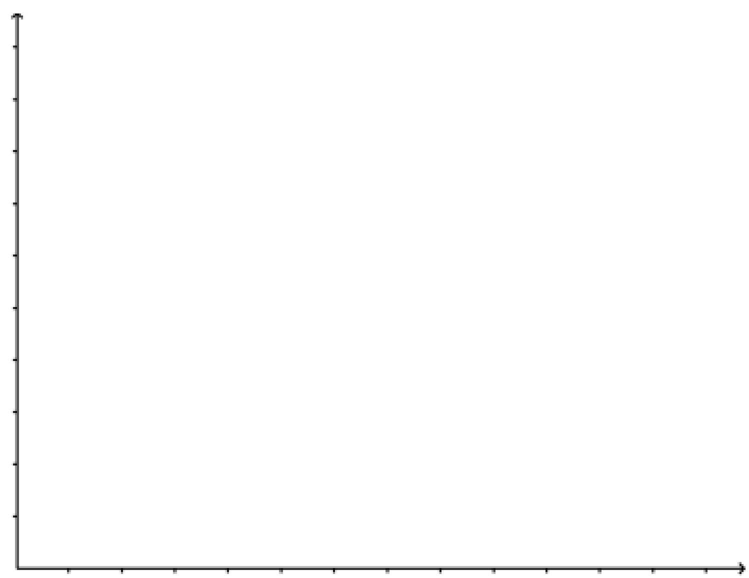
5. Many students like to eat microwave popcorn as they study for the ACT. Microwave popcorn producers assume that the time it takes for a kernel to pop is distributed normally with a mean of 120 seconds and a standard deviation of 13 seconds for a standard microwave oven. If you're a devoted popcorn studier, you don't want a lot of un-popped kernels, but you know that if you leave the bag in long enough to be sure that all the kernels are popped, some of the popcorn will burn. How much time would you recommend for microwaving the popcorn? Use a normal distribution curve and the features of a normal distribution to explain your answer.

6. While your teacher displays a clock that can display time in seconds, use the stop watch on your cell phone to determine how accurate **you can time** a five second interval. Repeat the data collection until you have 20 data samples. Combine your data with your group's data to create a frequency table and histogram and identify the mean and the standard deviation for both your data and your group's data. How accurate was your timing compared to your overall group's data?

My Data	**Group's Data**
Mean:	Mean:
Standard Deviation:	Standard Deviation

Frequency Table and Histogram:

Interval	Frequency



How accurate was your timing compared to your overall group's data?

8.3 Resource Page
Standard Normal Probabilities

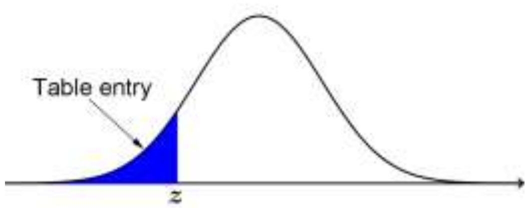


Table entry for z is the area under the standard normal curve to the left of z.

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
−3.4	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0002
−3.3	.0005	.0005	.0005	.0004	.0004	.0004	.0004	.0004	.0004	.0003
−3.2	.0007	.0007	.0006	.0006	.0006	.0006	.0006	.0005	.0005	.0005
−3.1	.0010	.0009	.0009	.0009	.0008	.0008	.0008	.0008	.0007	.0007
−3.0	.0013	.0013	.0013	.0012	.0012	.0011	.0011	.0011	.0010	.0010
−2.9	.0019	.0018	.0018	.0017	.0016	.0016	.0015	.0015	.0014	.0014
−2.8	.0026	.0025	.0024	.0023	.0023	.0022	.0021	.0021	.0020	.0019
−2.7	.0035	.0034	.0033	.0032	.0031	.0030	.0029	.0028	.0027	.0026
−2.6	.0047	.0045	.0044	.0043	.0041	.0040	.0039	.0038	.0037	.0036
−2.5	.0062	.0060	.0059	.0057	.0055	.0054	.0052	.0051	.0049	.0048
−2.4	.0082	.0080	.0078	.0075	.0073	.0071	.0069	.0068	.0066	.0064
−2.3	.0107	.0104	.0102	.0099	.0096	.0094	.0091	.0089	.0087	.0084
−2.2	.0139	.0136	.0132	.0129	.0125	.0122	.0119	.0116	.0113	.0110
−2.1	.0179	.0174	.0170	.0166	.0162	.0158	.0154	.0150	.0146	.0143
−2.0	.0228	.0222	.0217	.0212	.0207	.0202	.0197	.0192	.0188	.0183
−1.9	.0287	.0281	.0274	.0268	.0262	.0256	.0250	.0244	.0239	.0233
−1.8	.0359	.0351	.0344	.0336	.0329	.0322	.0314	.0307	.0301	.0294
−1.7	.0446	.0436	.0427	.0418	.0409	.0401	.0392	.0384	.0375	.0367
−1.6	.0548	.0537	.0526	.0516	.0505	.0495	.0485	.0475	.0465	.0455
−1.5	.0668	.0655	.0643	.0630	.0618	.0606	.0594	.0582	.0571	.0559
−1.4	.0808	.0793	.0778	.0764	.0749	.0735	.0721	.0708	.0694	.0681
−1.3	.0968	.0951	.0934	.0918	.0901	.0885	.0869	.0853	.0838	.0823
−1.2	.1151	.1131	.1112	.1093	.1075	.1056	.1038	.1020	.1003	.0985
−1.1	.1357	.1335	.1314	.1292	.1271	.1251	.1230	.1210	.1190	.1170
−1.0	.1587	.1562	.1539	.1515	.1492	.1469	.1446	.1423	.1401	.1379
−0.9	.1841	.1814	.1788	.1762	.1736	.1711	.1685	.1660	.1635	.1611
−0.8	.2119	.2090	.2061	.2033	.2005	.1977	.1949	.1922	.1894	.1867
−0.7	.2420	.2389	.2358	.2327	.2296	.2266	.2236	.2206	.2177	.2148
−0.6	.2743	.2709	.2676	.2643	.2611	.2578	.2546	.2514	.2483	.2451
−0.5	.3085	.3050	.3015	.2981	.2946	.2912	.2877	.2843	.2810	.2776
−0.4	.3446	.3409	.3372	.3336	.3300	.3264	.3228	.3192	.3156	.3121
−0.3	.3821	.3783	.3745	.3707	.3669	.3632	.3594	.3557	.3520	.3483
−0.2	.4207	.4168	.4129	.4090	.4052	.4013	.3974	.3936	.3897	.3859
−0.1	.4602	.4562	.4522	.4483	.4443	.4404	.4364	.4325	.4286	.4247
−0.0	.5000	.4960	.4920	.4880	.4840	.4801	.4761	.4721	.4681	.4641

8.3 Resource Page
Standard Normal Probabilities

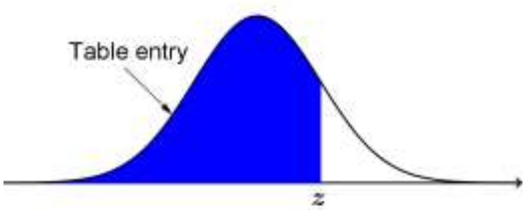


Table entry for *z* is the area under the standard normal curve to the left of *z*.

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	5239	.5279	5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998

8.3 Warm Up
Y B Normal? – *A Solidify Understanding Task*

As a college admissions officer, you get to evaluate hundreds of applications from students that want to attend your school. Many of them have good grades, have participated in school activities, have done service within their communities, and all kinds of other attributes that would make them great candidates. One part of the application that is carefully considered is the applicant's score on the college entrance examination. At the college you work for, some students took the ACT and others took the SAT.

You have to make a final decision on two applicants. They are both wonderful students with the very same G.P.A. and class rankings. It all comes down to their test scores. The first student, Clarita, took the ACT and received a score of 29 in mathematics. The second student, Carlos, took the SAT and received a score of 680 in mathematics. Since you are an expert in college entrance exams, you know that both tests are designed to be normally distributed. A perfect score on the ACT is 36. The ACT mathematics section has a mean of 21 and standard deviation of 5.2. (Source: National Center for Education Statistics 2010) A perfect score on the SAT math section is 800. The SAT mathematics section has a mean of 516 and a standard deviation of 116. (Source: www.collegeboard.com 2010 Profile).

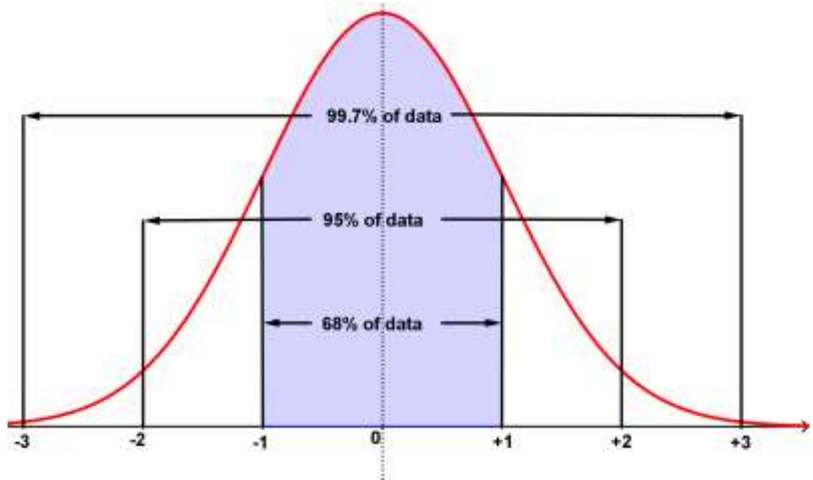
1. Based only on their test scores, which student would you choose and why?

This analysis is starting to make you hungry, so you call your friend in the Statistics Department at the university and ask her to go to lunch with you. During lunch, you tell her of your dilemma. The conversation goes something like this:

You: I'm not sure that I'm making the right decision about which of two students to admit to the university. Their entrance exam scores seem like they're in about the same part of the distribution, but I don't know which one is better. It's like trying to figure out which bag of fruit weighs more when one is measured in kilograms and one is measured in pounds. They might look like about the same amount, but you can't tell the exact difference unless you put them on the same scale or convert them to the same units.

Statistician: Actually, there is a way to make comparisons on two different normal distributions that is like converting the scores to the same unit. The scale is called the **"standard normal distribution."** It was invented so normal distributions can be compared. It is set up so that the mean is 0 and the standard deviation is 1.

Here's what your statistician friend drew on her napkin to show you the standard normal distribution:



You: Well, that looks just like the way I always think of normal distributions.

Statistician: Yes, it's pretty simple. When we use this scale, we give things a **z-score**. A z-score of 1 means that it's 1 standard deviation above the mean. A z-score of -1.3 means that it is between 1 and 2 standard deviations below the mean.

What's even better is that when we have a z-score, there are tables that will show the area under the curve to the left of that score. For a test score like the ACT or SAT, it shows the percentage of the population (or sample) that is below that score. I've got a z-score table right here in my purse. See, the z-score is -1.3, then 9.68% of the population scored less. You can also say that 90.32% of the population scored better, so -1.3 wouldn't be a very good score on a test.

Try it: Let's say you had two imaginary test takers, Jack and Jill. Jack's z-score was 1.49 and Jill's z-score was 0.89.

2. What percent of the test takers scored below Jack? What percent scored above Jack?

3. What percent of the test takers scored below Jill? What percent scored above Jill?

4. What percent of the test takers scored between Jack and Jill?

5. Jack and Jill's friend, Jason, scored -1.49. Find the number of test takers that scored above him without using a table or technology. Explain your strategy.

You: That's very cool, but the two scores I'm working with are not given as z-scores. Is there some way that I can transform values from a normal distribution, like the scores on the ACT or SAT, to z-scores?

Statistician: Sure. The scale wouldn't be so amazing if you couldn't use it for any normal distribution. There's a little formula for transforming a data point from any normal distribution to a standard normal distribution:

$$\mathbf{z-score} = \frac{\mathbf{data\ point - mean}}{\mathbf{standard\ deviation}}$$

6. The mean score on the ACT is 21 and the standard deviation is 5.2. Use the formula to figure out the z-score. Explain why this value is reasonable for Clarita's ACT score of 29.

You: That's great. I'm going back to the office to decide which student is admitted.

7. Compare the scores of Clarita and Carlos. Explain which student has the highest mathematics test score and why.

Practice Problems:

The notation $\boldsymbol{N(\mu, \sigma)}$ represents a normal distribution where the **mean** ($\boldsymbol{\mu}$) and the **standard deviation** ($\boldsymbol{\sigma}$) are given.

1. A customer calling a call center spends an average of 45 minutes on hold during peak season, with a standard deviation of 12 minutes. Suppose these times are normally distributed. This information can be denoted as $N(45, 12)$. Find the probability that customer will be on hold for each interval of time:

   a. $P(x > 54)$

   b. $P(x < 24)$

   c. $P(24 < x < 54)$

   d. $P(x > 39)$

2. Find each probability for the given normal distribution.

   a. $N(5, 4)$, find $P(x > 8)$

   b. $N(3, 2)$, find $P(1.2 < x < 4.5)$

   c. $N(75, 12)$, find $P(80 < x < 100)$

8.3 Whoa! That's Weird!
A Practice Understanding Task

Each of the stories below is based upon normal distributions. Rank order these stories from **most unusual to least unusual**. (1 is the most unusual, 6 is the least unusual.) In each case, explain your ranking.

A. The number of red loops in a box of Tutti-Frutti-O's is normally distributed with mean of 800 loops and standard deviation 120. Tony bought a new box, opened it, and counted 1243 red loops.

Rank______ Explanation:____

B. The weight of house cats is normally distributed with a mean of 10 pounds and standard deviation 2.1 pounds. My cat, Big Boy, weighs 6 pounds.

Rank______ Explanation:____

C. The lifetime of a battery is normally distributed with a mean life of 40 hours and a standard deviation of 1.2 hours. I just bought a battery and it died after just 20 hours

Rank______ Explanation:____

D. The amount that a human fingernail grows in a year is normally distributed with a mean growth of 3.5 cm and a standard deviation of 0.63 cm. My neighbor's thumbnail grew all year without breaking and it is 4.6 cm long.

Rank______Explanation:____

E. My little brother was digging in the garden and found a giant earthworm that was 35 cm long. The length of earthworms is normally distributed with a mean length of 14 cm and a standard deviation of 5.3 cm.

Rank______Explanation:____

F. The mean length of a human pregnancy is 268 days with a standard deviation of 16 days. My aunt just had a premature baby delivered after only 245 days.

Rank______Explanation:____

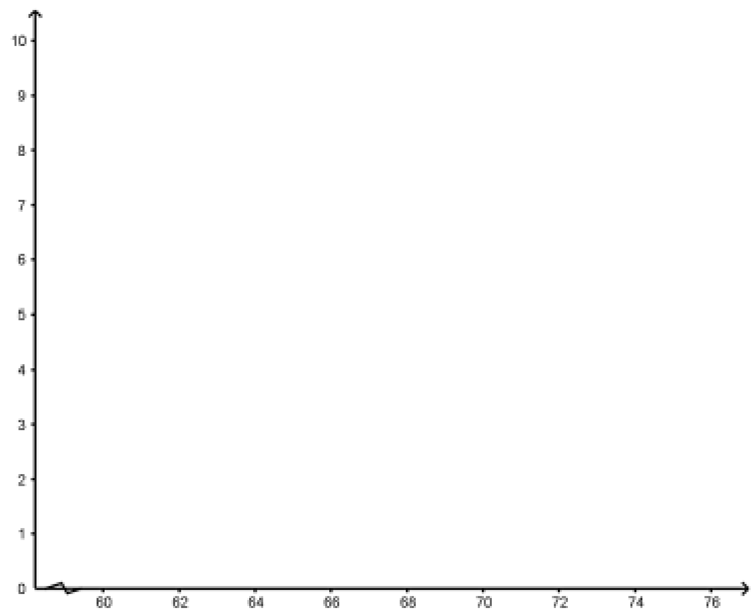
8.4 Warm Up
Comparing Samples

You are performing a study about the height of 20- to 29- year-old men. A previous study found the height to be normally distributed, with a mean of 69.2 inches and a standard deviation of 2.9 inches. You randomly sample 30 men and find their heights to be as follows. (Source: National Center for Health Statistics)

72.1, 71.2, 67.9, 67.3, 69.5, 68.6, 68.8, 69.4, 73.5, 67.1, 69.2, 75.7, 71.1, 69.6, 70.7, 66.9, 71.4, 62.9, 69.2, 64.9, 68.2, 65.2, 69.7, 72.2, 67.5, 66.6, 66.5, 64.2, 65.4, 70.0

1. Complete the frequency table and draw a frequency histogram to display these data. Is it reasonable to assume that the heights are normally distributed? Why?

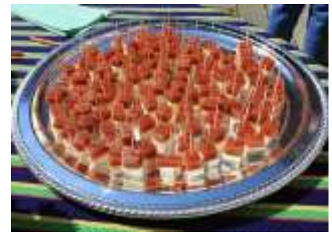
Interval of Heights	Frequency
$62-<64$	
$64-<66$	
$66-<68$	
$68-<70$	
$70-<72$	
$72-<74$	
$74-<76$	



2. Find the mean and standard deviation of your sample.

3. Compare the mean and standard deviation of your sample with those in the previous study. Discuss the differences.

8.4 Would You Like to Try a Sample?
A Develop Understanding Task



In the task *Whoa! That's Weird!,* you saw a number of statistics for things like the average weight of a house cat. You know it would be impossible to measure all the house cats in the world to find their average weights, but scientists still claim to know it.

You've probably heard it many times before: "Survey results show that 54% of Americans believe that. . ." You're sure that you didn't participate in the survey and neither did anyone you know, and yet, the researchers claim that the survey represents the beliefs of all Americans.

How can this be possible? In the next few tasks, we'll explore how statistics allow us to draw conclusions about an entire group without actually working with the entire group. Sometimes the results make sense and other times you might think that they just can't be right. We will learn how to make judgments about statistical studies, based on the methods that have been used.

First, we need to get our vocabulary straight. A **population** refers to the entire group that we are interested in. A **sample** is a subset of the population; a group selected to represent the entire population. The **parameter of interest** is the question we want answered about the population.

For each of the scenarios below, identify the population, the sample, and the parameter of interest.

1. A grocery store wants to know the average number of items that shoppers purchase in each visit to the store. They decide to count the items in the cart for every twentieth person through the check stand.

   Population ____

   Sample ____

   Parameter of interest ____

2. A team of biologist wants to know the average weight of fish in a lake. They decide to drop a net and measure all the fish caught in three different locations in the lake.

   Population ____

   Sample ____

   Parameter of interest ____

3. There are lots of different ways that a sample can be chosen from a population.

   a. Group the following examples of ways to select a sample into two categories: **random and non-random.**

   A. You are in charge of school activities. You want to know what activities students would prefer to participate in during the school year. You decide to put the name of each student in the school into a big bowl. You draw 100 names and ask those students to respond to a survey about the activities they prefer.

   B. You are in charge of school activities. You want to know what activities students would prefer to participate in during the school year. You assign each student in the school a number. You randomly select a starting number between 1 and 10 and then select every tenth student in the list from that point forward.

   C. You are in charge of school activities. You want to know what activities students would prefer to participate in during the school year. You use the rosters from each homeroom class. You put the all the names from one class into the bowl and draw two names from the class. You go through each homeroom class, drawing 2 names from each class. You ask those students to respond to a survey about the activities they prefer.

   D. You are in charge of school activities. You want to know what activities students would prefer to participate in during the school year. You get the list of all the homeroom classes and randomly select 5 classes. You go to each of the classes selected and survey all the students in that class.

   E. You are in charge of school activities. You want to know what activities students would prefer to participate in during the school year. You stand in the cafeteria during your lunch break and ask students if they would be willing to participate in your survey as they walk by.

   F. You are in charge of school activities. You want to know what activities students would prefer to participate in during the school year. You make a lot of copies of the survey about the activities that students prefer and you put them on a table outside the cafeteria. Students can choose to take the survey and drop their responses into a big box on the table.

   G. You are interested in finding out the percent of residents in the city that have experienced a robbery in the past year. Using the city property records, you assign each residence a number. You use a random number generator to give you a list of numbers. You contact the residence that corresponds to that number to ask your questions.

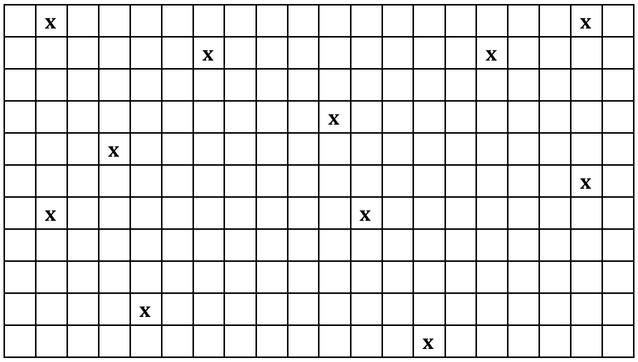
   H. You want to know the average number of hours that high school seniors spend playing video games in your state. You randomly select 20 high schools in the state and then ask all the seniors at each of the 20 high schools about their video game habits.

   I. An auto analyst is conducting a satisfaction survey, sampling from a list of 10,000 new car buyers. The list includes 2,500 Ford buyers, 2,500 GM buyers, 2,500 Honda buyers, and 2,500 Toyota buyers. The analyst selects a sample of 400 car buyers, by randomly sampling 100 buyers of each brand.

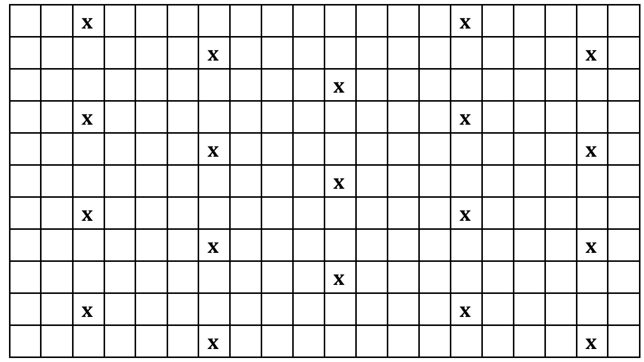
   J. A shopping mall management company would like to know the average amount of money that shoppers in the mall spend during their visit. They post two survey takers near one of the exits who ask shoppers to tell them how much they spent as they leave the mall.

   K. A restaurant owner wants to find out the average number of dishes ordered at each table served on Friday evenings, their busiest time. She decides to collect and analyze every fifth receipt of the night, starting at 6:00 p.m.

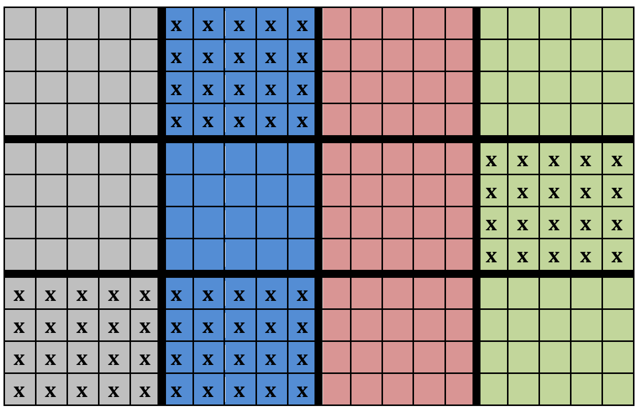
L.



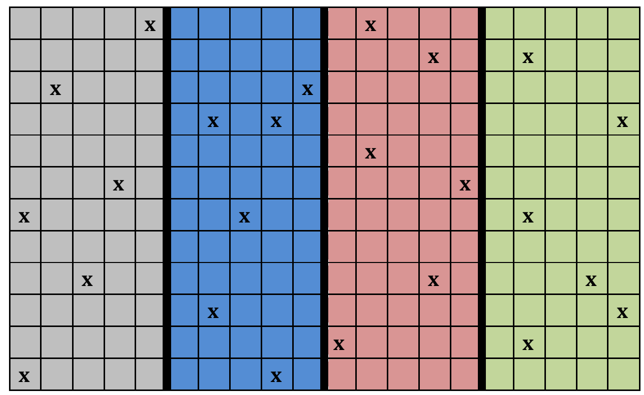
M.



N.



O.



b. Now that you have a random group and a non-random group, sort these further into six groups based on the following titles:

Simple random sample
Systematic random sample
Stratified random sample
Cluster random sample
Convenience sample
Volunteer sample

4. What might be some of the advantages and disadvantages of each type of sampling method?

5. A person you know owns a small theater that shows local dramatic productions. She wants to know the average age of the people that buy tickets so that she can better select which plays to stage. Explain to the owner why selecting the first 20 people that arrive for the show may not be a representative sample.

6. Describe a process for selecting a representative sample of the theater patrons described in question 5.

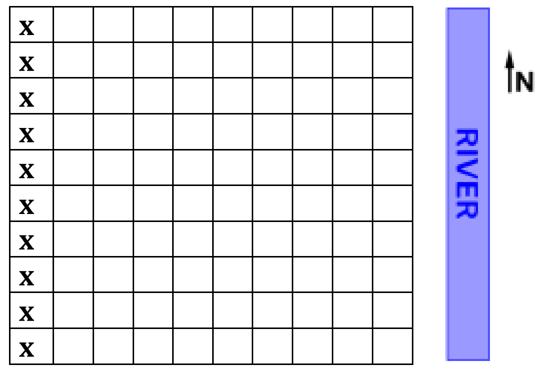
8.5 Rolling Down the River
A Solidify Understanding Task

https://upload.wikimedia.org/wikipedia/common

A farmer has just cleared a new field for corn. It is a unique plot of land in that a river runs along one side. The corn looks good in some areas of the field but not others. The farmer is not sure that harvesting the field is worth the expense. He has decided to harvest 10 plots and use this information to estimate the total yield. Based on this estimate, he will decide whether to harvest the remaining plots.

Method Number One: Convenience Sample

The farmer began by choosing 10 plots that would be easy to harvest. They are marked on the grid at right:

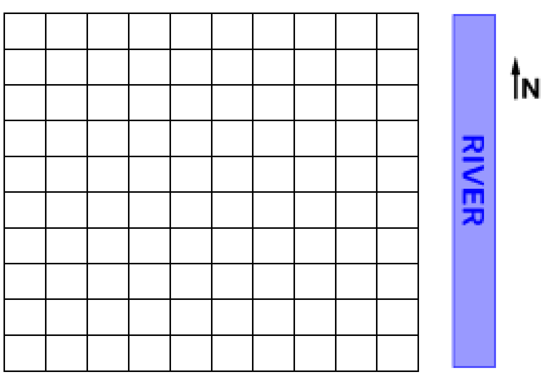


Since then, the farmer has had second thoughts about this selection and has decided to come to you (knowing that you are somewhat knowledgeable, but far cheaper than a professional statistician) to determine the approximate yield of the field.

You will still be allowed to pick 10 plots to harvest early. Your job is to determine which of the following methods of sampling is the best to use – and to decide if this is an improvement over the farmer's original plan.

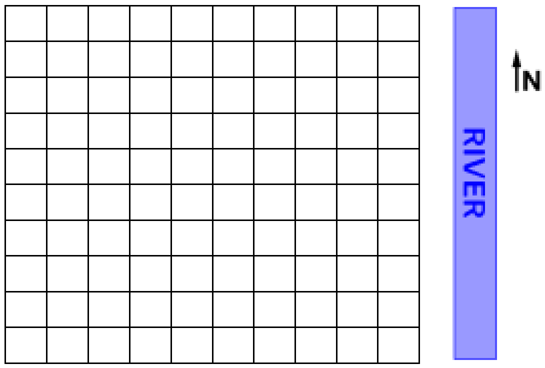
Method Number Two: Simple Random Sample

Use technology or the random number table at the end of the task to choose 10 plots to harvest. Mark them on the grid below and describe your method of selection. (Label your rows 0 through 9, columns 0 through 9.)



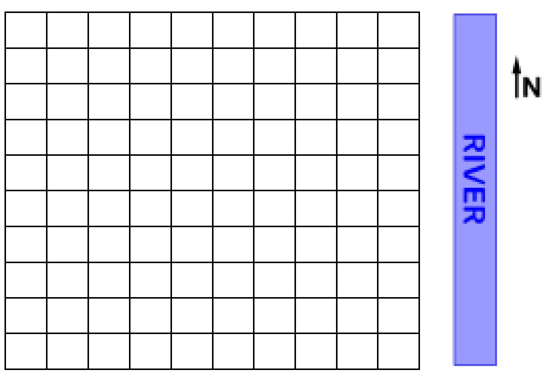
Method Number Three: Stratified Sample

You and the farmer think the river might have a strong influence on corn production so you decide to consider the field as grouped in vertical columns (called strata – remember that you can only stratify your sample when you think a factor will have a strong influence on the outcome). Randomly choose one plot from each **vertical column** and mark on the grid. (Label your **columns A through J**, rows 0 through 9.)



Method Number Four: Stratified Sample (Horizontal)

You and the farmer rethink the plan and decide that direction (north – south) may have a strong influence on corn production. You consider the field as grouped in horizontal rows (also called strata). Randomly choose one plot from each **horizontal row** and mark them on the grid. (Label your **rows A through J**, columns 0 through 9.)



OK, the crop is ready. Use the grid on the resource page provided by your teacher to estimate the **mean** yield per plot based on each of the four sampling techniques (Add the yield for each of the 10 plots harvested and divide the total by 10). To complete the "estimate the total yield" column, multiply the "mean yield per plot" values by 100.

Sampling Method	Mean Yield Per Plot	Estimate of Total Yield
Convenience Sample (farmer's)		
Simple Random Sample		
Vertical Strata		
Horizontal Strata		

Submit your data to your teacher before completing the observation questions below.

Observations:

1. You have looked at four different methods of choosing plots. Is there a reason to choose one method over another?

2. How did your estimates vary according to the different sampling methods you used?

3. Compare your results to someone else in the class. Were your results similar?

4. Using the combined class data that your teacher collected, make a boxplot for each sampling method (except the convenience sample). Compare and contrast the three box plots. What do you see?

5. Which sampling method should you use? Why do you think this method is best?

6. What was the actual yield of the farmer's field? How did the boxplots relate to this actual value?

TABLE OF RANDOM SAMPLING NUMBERS

Row #											
1	46573	25595	85393	30995	89198	27982	53402	93965	34095	52666	19174
2	48360	22527	97265	76393	64809	15179	24830	49340	32081	30680	19655
3	93093	06243	61680	07856	16376	39440	53537	71341	57004	00849	74917
4	39975	81837	16656	06121	91782	60468	81305	49684	60672	14110	06927
5	06907	11008	42751	27756	53498	18602	70659	90655	15053	21916	81825
6	72905	56420	69994	98872	31016	71194	18738	44013	48840	63213	21069
7	91977	05463	07972	18876	20922	94595	56869	69014	60045	18425	84903
8	14342	63661	10281	74553	18103	57740	84378	25331	12566	58678	44947
9	36857	53342	53988	53060	59533	38867	62300	08158	17983	16439	11458
10	69578	88231	33276	70997	79936	56865	05859	90106	31595	01547	85590
11	40961	48235	03427	49626	69445	18663	72695	52180	20847	12234	90511
12	93969	52636	92737	88974	33488	36320	17617	30015	08272	84115	27156
13	61129	87529	85689	48237	52267	67689	93394	01511	26358	85104	20285
14	97336	71048	08178	77233	13916	47564	81056	97735	85977	29372	74461
15	12765	51821	51259	77452	16308	60756	92144	49442	53900	70960	63990
16	21382	52404	60268	89368	19885	55322	44819	01188	65255	64835	44919
17	54092	33362	94904	31273	04146	18594	29852	71585	85030	51132	01915
18	53916	46369	58586	23216	14513	83149	98736	23495	64350	94738	17752
19	97628	33787	09998	42698	06691	76988	13602	51851	46104	88916	19509
20	91245	85828	14346	09172	30168	90229	04734	59193	22178	30421	61611
21	58492	22421	74103	47070	25306	76468	26384	58151	06646	21524	15227
22	32363	05597	24200	13363	38005	94342	28728	35806	06912	17012	64161

Row #											
23	27001	87637	87308	58731	00256	05834	15398	46557	41135	10367	07684
24	33062	28834	08751	19731	92420	60952	61280	50001	67658	32586	86679
25	72295	04839	96423	24878	82651	66566	14778	76797	14780	13300	87074
26	20591	68086	26432	46901	20849	89768	81536	86645	12659	92259	57102
27	57392	39064	66432	84673	40027	32832	61362	98947	96067	64760	64584
28	04213	25669	26422	44407	44048	37937	63904	45766	66134	75470	66520
29	24618	64117	94305	26766	25940	39972	22209	71500	64568	91402	42416
30	04711	87917	77341	42206	35126	74087	99547	81817	42607	43808	76655
31	69884	62797	56170	86324	88072	76222	36086	84637	93161	76038	65855
32	65795	95876	55293	18988	27354	26575	08625	40801	59920	29841	80150
33	57948	29888	88604	67917	48708	18912	82271	65424	69774	33611	54262
34	83473	73577	12908	30883	18317	28290	35797	05998	41688	34952	37888
35	42595	27958	30134	04024	86385	29880	99730	55536	84855	29080	09250
36	56349	90999	49127	20044	59931	06115	20542	18059	02008	73708	83517

8.6 Warm Up
Let's Investigate – A Solidify Understanding Task

Part I:
When we want to draw conclusions about some population, there are at least two different statistical ideas to consider. In the task *Would You Like to Try a Sample?*, we learned that it is usually more practical to sample the population rather than measure everyone or everything in the population.

The second thing to consider is how to measure the parameter of interest, the thing we want to know about the population. Sometimes it's obvious, like if you want to know the average weight of a population, you determine a sample and then put each of the subjects on a scale. Three other techniques are:

- **Surveys:** When researchers want to know how people feel, what their preferences are, what they own, how much money they make, etc., they often construct a survey to ask the people in the sample about the parameter of interest.
- **Observational Studies:** In this type of study, researchers observe the behavior of the participants/subjects without trying to influence them in any way so they can learn about the parameter of interest.
- **Experiments:** In an experiment, researchers manipulate the variables to try to determine cause and effect.

1. Imagine that you want to know whether a new diet plan is effective in helping people lose weight. You might choose any of the three methods to determine this.

   - If you used a **survey**, you could simply ask people that had tried the diet plan if they lost weight.
   - If you used an **observational study**, you might monitor volunteers that try the diet plan and measure how much weight they lost.
   - If you used an **experiment**, you might randomly assign participants to two groups. One group (*the control group*) eats as they normally would and the other group (*the experimental group*) eats according to the diet plan. At the end of two months, the two groups are compared to see the average weight gain or loss in each group.

   Based on these three examples,

   a. What are some possible advantages and disadvantages of **surveys**?

   b. What are some possible advantages and disadvantages of **observational studies**?

   c. What are some possible advantages and disadvantage of **experiments**?

2. Identify which method is illustrated by each example:

    a. To determine whether drinking orange juice prevents colds, researchers randomly assigned participants to a group that drank no orange juice or to a group that drank two glasses of orange juice daily. Researchers measured the number of colds that each group had over the course of the year and compared the results of the two groups.

    b. To determine whether exercise reduces the number of headaches, researchers randomly selected a group of participants and recorded the number of hours each participant exercised and the number of headaches each participant experienced.

    c. To determine the effectiveness of a new advertising campaign, a restaurant asked every tenth customer if they had seen the advertisement and if it had influenced their decision to visit the restaurant.

    d. To determine if a new drug is an effective treatment for the flu, researchers randomly selected two groups of people that had the flu. One group was given a placebo (a sugar pill that has no physical effect) and one group was given the new drug. Researchers measured the number of days that participants experienced flu symptoms and compared the two groups to see if they were different.

    e. To determine if higher speed limits cause more traffic fatalities, researchers compared the number of traffic deaths on randomly selected stretches of highway with 65 mph speed limits to the number of traffic deaths on an equal number of randomly selected stretches of highway with 75 mph speed limits.

Part II:

3. Create a parameter of interest and describe how you might select a sample and use a **survey** to investigate.

4. Create a parameter of interest and describe how you might select a sample and use an **observational study** to investigate.

5. Create a parameter of interest and describe how you might select a sample and use an **experiment** to investigate.

6. Describe the method you would use to determine if excessive texting causes bad grades. Explain why you chose that method and what conclusions could be drawn from the study.

8.6 Slacker's Simulation
A Solidify Understanding Task

I know a student who forgot about the upcoming history test and did not study at all. To protect his identity, I'll just call him Slacker. When I reminded Slacker that we had a test in the next class, he said that he wasn't worried because the test has 10 true/false questions. Slacker said that he would totally guess on every question, and since he's always lucky, he thinks he will get at least 8 out of 10 correct.

I'm skeptical, but Slacker said, "Hey, sometimes you flip a coin and it seems like you just keep getting heads. You may only have a 50/50 chance of getting heads, but you still might get heads several times in a row. I think this is just about the same thing. It could be my lucky day!"

1. What do you think of Slacker's claim? Is it possible for him to get 8 out of 10 questions right? Explain.

I thought about it for a minute and said, "Slacker, I think you're on to something. I'm not sure that you will get 80% on the test, but I agree that the situation is just like a coin flip. It's either one way or the other and they are both equally likely if you're just guessing." My idea is to use a coin flip to simulate the T/F test situation. We can try it many times and see how often we get 8 out of 10 questions right. I'm going to say that if the coin lands on heads, then you guessed the problem correctly. If it lands on tails, then you got it wrong.

2. Try it a few times yourself. To save a little time, just flip 10 coins at once and count up the number of heads for each test.

	# Correct (Heads)	**# Incorrect (Tails)**	**% Correct**
Test 1			
Test 2			
Test 3			
Test 4			
Test 5			

3. Did you get 8 out of 10 correct in any of your trials?

4. Based on your trials, do you think Slacker has a good chance of getting 80% correct?

5. Go to the website http://www.shodor.org/interactive/activities/Coin/ to simulate 50 more tests. Now what do you think of Slacker's chances of getting 80% correct? Explain.

Integrated Math 3
Module 8
Statistics
Ready, Set, Go! Homework

Adapted from

The Mathematics Vision Project:
Scott Hendrickson, Joleigh Honey, Barbara Kuehl,
Travis Lemon, Janet Sutorius

Name **Statistics | 8.1**

Ready, Set, Go!

Ready

Topic: Standard deviations, percentiles

1. Jordan scores a 53 on his math test. The class average is 57 with a standard deviation of 2 points. How many standard deviations below the mean did Jordan score?

2. In Jordan's science class, he scored a 114. The class average was a 126 with a standard deviation of 6 points. How many standard deviations below the mean did Jordan score?

3. Rank the data sets below in order of greatest standard deviation to smallest:

   $A = \{1, 2, 3, 4\}$ $B = \{2, 2, 2, 2\}$ $C = \{2, 4, 6, 8\}$ $D = \{4, 5, 6, 8\}$ $E = \{1, 1.5, 2, 2.5\}$

4. Robin made it to the swimming finals for her state championship meet. The times in the finals were as follows:

   {2:10.3, 2:12.5, 2:12.38, 2:12.7, 2:20.45, 2:21.43}

   If Robin's time was a 2:12.7, what percent of her competitors did she beat?

5. In statistics, μ is the symbol for **mean** and σ is the symbol for **standard deviation**. Using technology (i.e. a graphing calculator or the website http://www.miniwebtool.com/population-standard-deviation-calculator/), identify the mean and standard deviation for the data set below. Round to the nearest hundredth.

   {1.23, 1.3, 1.1, 1.48, 1, 1.14, 5.21, 5.1, 4.63}

   $\mu =$ $\sigma =$

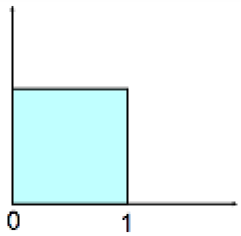
6. For the data in number 5, what value is one standard deviation above the mean?

   Three standard deviations below the mean?

Set

Topic: Properties of normal curves

7. **For each distribution, identify the properties that match with a *normal distribution*. Then decide if the distribution is normal or not.**

<table>
<tr>
<td>A.

</td>
<td>Normal Properties:

Normal? Yes or No</td>
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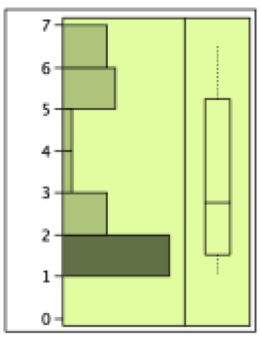
<td>B.

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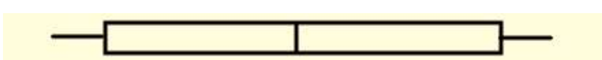
<td>C.
</td>
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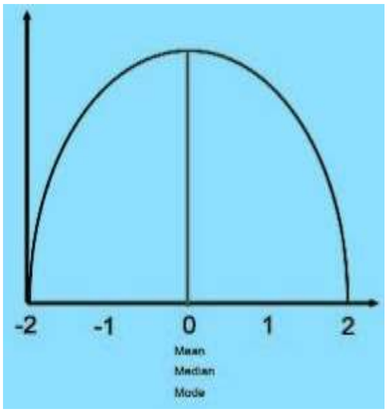
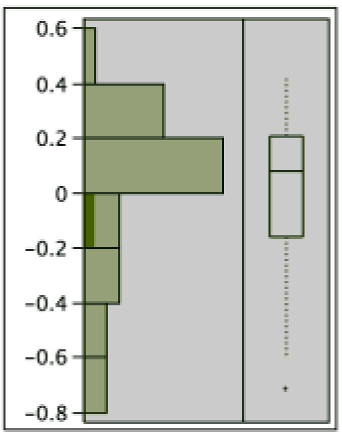
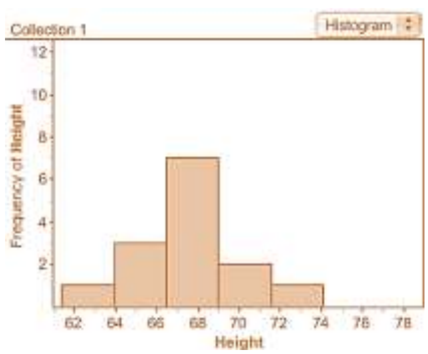
D. 	Normal Properties: Normal? Yes or No
E.  **Mean = 0 Median = 0.1 Mode = 0.1**	Normal Properties: Normal? Yes or No
F.  **Mean = 68 Median = 68 Mode = 68**	Normal Properties: Normal? Yes or No

8. If two normal distributions have the same standard deviation, 4.9, but different means (3 and 6), how will the two normal curves look in relation to each other? Draw a sketch of each normal curve below.

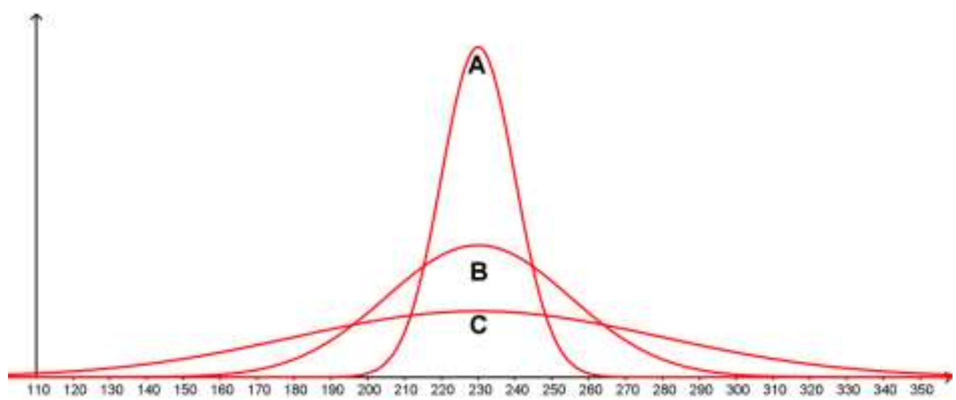
9. If two normal distributions have the same mean, 3, but different standard deviations (1 and 4), how will they look in relation to each other? Draw a sketch of each normal curve below.

10. Several normal curves are given below. Estimate the standard deviation of each one using the mean and location of the inflection point.

A: ________

B: ________

C: ________



Go

Topic: Inverses

Write the inverse of the given function. Keep the answer in the same format as the problem:

11. $f(x) = 3x^2 + 2$

12. $g(x) = \frac{2x-7}{4}$

13. $h(x) = 3 + \sqrt{2x - 1}$

14.

x	y
12	24
14	38
−7	4
13	6
7	0

x	y^{-1}

Determine if the following functions are inverses by finding $f\big(g(x)\big)$ and $g\big(f(x)\big)$

15. $f(x) = 2x + 3$ and $g(x) = \frac{1}{2}x - \frac{3}{2}$

16. $f(x) = 2x^2 - 3$ and $g(x) = \sqrt{\frac{x^2}{2} + 3}$

Ready, Set, Go!

Ready

Topic: Law of large numbers

1. You and your friend are rolling one die over and over again. After 6 rolls, your friend has rolled four fives. Are you surprised by these results? Explain

2. After rolling the die 50 times, you now notice that you rolled a total of 20 fives. Are you surprised now? Explain.

3. You survey 100 people in your school and ask them if they feel your school has adequate parking. Only 30% of the sample feels the school has enough parking. If you have 728 students total in your school, how many would you expect out of all the student body that felt there was enough parking?

Set

Topic: Normal curves

4. The population of NBA players heights is normally distributed with a mean of 6'7" and a standard deviation of 3.9 inches. (http://www.wikipedia.org) Greg is considered unusually tall for his high school at 6' 2".

   a. What percent of NBA players are taller than Greg?

   b. What percent are shorter?

   c. How tall would Greg have to be in order to be in the top 2.5% of NBA player heights?

5. The average height of boys at Greg's school is 5'6" with a standard deviation of 2" (reminder: Greg is 6'2" tall). If we assume the population is normal…

   a. What percent of students in Greg's school is he taller than?

   b. What percent of students are between 5' and 5'8"?

6. Jordan is drinking a cup of hot chocolate. From previous research, he knows that it takes a cup of hot chocolate 10 minutes to reach a temperature where his tongue will not burn. The time it takes the chocolate to cool varies normally with a standard deviation of 2 minutes.

   a. How long should Jordan wait to drink his hot chocolate if he wants to be 84% sure that he won't burn himself?

   b. If Jordan waits 8 minutes, what percent of the time will he not burn his tongue?

Go

Topic: Logarithms

Use the properties of logarithms to expand the expression as a sum or difference, and/or constant multiple of logarithms. Assume all variables are positive.

7. $\log_2 3x$

8. $\log_x \frac{5}{7}$

9. $\log \frac{x^2 2y^4}{3z^2}$

10. $\log_3 \frac{16x^2-36}{x^2}$

11. $\log \frac{x^2+12x+20}{5}$

12. $\log 10^5 \sqrt{y}$

Name **Statistics | 8.3**

Ready, Set, Go!

Ready

Topic: Probability

At Cardiff Kook Academy, there are 2500 students attending. Mariana surveys 40 of her friends on where they prefer to eat lunch. She created the following two-way table showing her results:

	9th Grade	**10th Grade**	**11th Grade**	**12th Grade**	**Totals**
School Cafeteria	18	6	2	1	**28**
Off Campus	2	4	3	4	**12**
Totals	**20**	**10**	**5**	**5**	**40**

Mariana plans to use her data to answer the following questions:

I. Do students, overall, prefer to eat on campus or off campus?
II. Is there a difference between grade levels for where students prefer to eat lunch?

1. In Mariana's sample, what percent of students prefer to eat in the school cafeteria? What percent prefer to eat off campus?

2. For each grade level in her sample, determine the percent of students that prefer to eat in the school cafeteria and the percent that prefer to eat off campus. Do you notice anything unusual?

3. Based on her sample, Mariana concludes that, overall, students at Cardiff Kook Academy like to eat lunch in the school cafeteria. Do you agree or disagree? Why?

Set

Topic: z-scores

A company makes a mean monthly income of $20,300 with a standard deviation of $3,200. In one given month, the company makes $29,500.

4. Find the z-score.

5. Assuming the company's monthly income follows a normal distribution, what percent of the time does the company make more than this amount? Less than this amount?

6. What percent of the time does the company make between $15,000 and $25,000?

7. If the company needs to make $16,400 in order to break even, how likely in a given month is the company to make a profit?

On the Wechsler Adult Intelligence Scale, an average IQ is 100 with a standard deviation of 15 units. (Source: http://en.wikipedia.org/wiki/Intelligence_quotient)

8. IQ scores between 90 and 109 are considered average. Assuming IQ scores follow a normal distribution, what percent of people are considered average?

9. One measure of "genius" is an IQ score of above 135. What percent of people are considered "genius"?

10. Einstein had an IQ score of 160. What was his z-score?

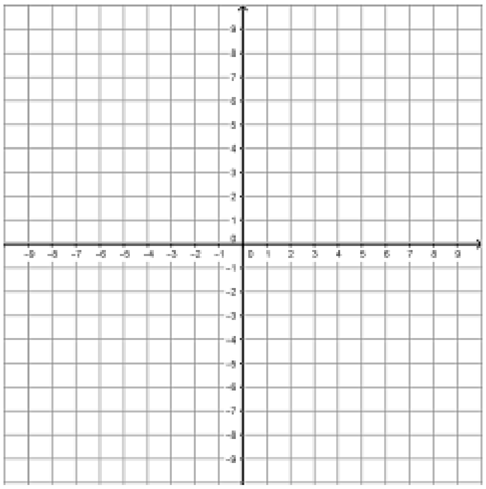
11. What is the probability of an individual having a higher IQ than Einstein?

Go

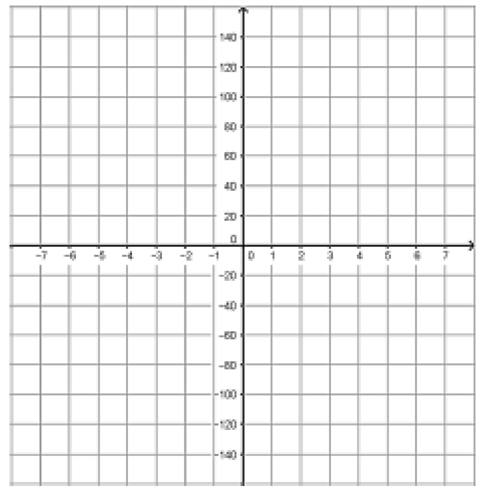
Topic: Sketching polynomials

Without using technology, sketch the graph of a polynomial function with the given characteristics. Explain how you know the graph of $f(x)$ looks like this.

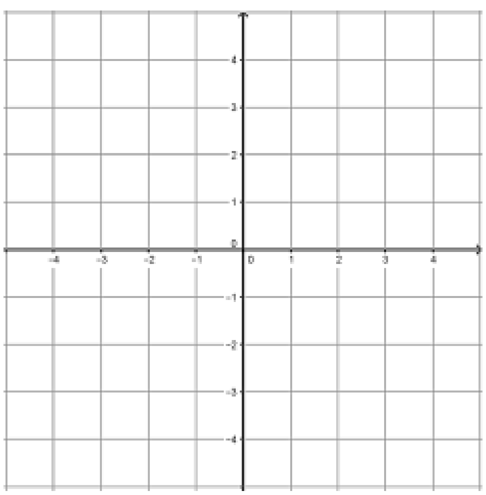
12. A quartic function with a leading coefficient of -2 with one double zero and two complex roots.



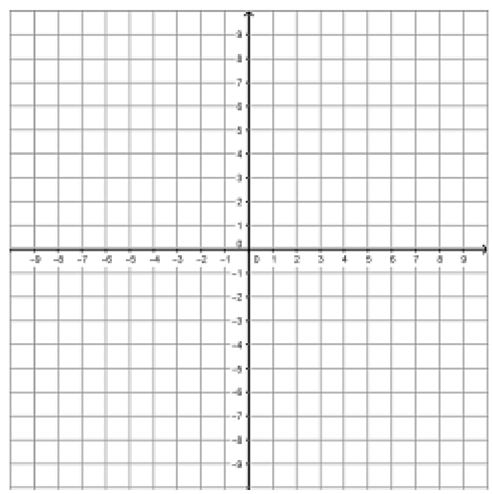
13. $f(x) = (x + 2)^2(x - 3)^3$



14. $g(x) = -(x + 2)^2(2x - 1)(x + 1)^3$



15. A cubic function with a leading coefficient of 4 and three positive roots.



Name **Statistics | 8.4**

Ready, Set, Go!

Ready

Topic: Causation

When collecting data, statisticians are often interested in making predictions. Sometimes, statisticians simply want to know if one variable is **correlated** with another variable. Often times, statisticians want to determine if one variable actually **causes** a change in another variable.

Given the examples below, decide whether you think the variables are correlated with each other or if one variable causes the other to change.

1. As the amount of food Ollie the elephant eats increases her weight also increases.

2. As Popsicle sales go up in the summer, the number of drownings also increases.

3. As Erika's feet grow longer, she grows taller.

4. As Tabatha gets older, her reading score improves in school.

Set

Topic: Population vs sample

For the following scenarios, identify the **population**, **sample** and **parameter of interest**.

5. The local school board wants to get parents to evaluate teachers. They select 100 parents and find that 89% approve of their child's teacher.

   Population: Sample: Parameter:

6. Jarret wants to know the average height of the students in his school. There are 753 students in his high school; he finds the heights of 52% of them.

   Population: Sample: Parameter:

7. A government official is interested in the percent of people at JFK airport that are searched by security. He watches 300 people go through security and observes 42 that are searched.

   Population: Sample: Parameter:

Topic: Types of samples

For each scenario, identify what type of sampling was used to obtain the sample. Explain whether or not you think the sample will be representative of the population it was sampled from.

8. Elvira surveys the first 60 students in the lunch line to determine if students at the school are satisfied with school lunch.

   Type of sample:

   Representative? Explain.

9. Elvira selects every 5th student in the lunch line to determine if students at the school are satisfied with school lunch.

   Type of sample:

   Representative? Explain.

10. Elvira randomly selects 7 different tables in the lunchroom and surveys every student at the table to determine if students at the school are satisfied with school lunch.

    Type of sample:

    Representative? Explain.

11. Elvira assigns every student in the school a number and randomly selects 60 students to survey to determine if students at the school are satisfied with school lunch.

    Type of sample:

    Representative? Explain.

12. Elvira wants to determine if students are satisfied with school lunch. She leaves surveys on a table for students to answer as the walk by.

    Type of sample:

    Representative? Explain.

13. Elvira wants to determine if students are satisfied with school lunch. She wants to include input from each grade level at the high school. She randomly surveys 25 freshman, 25 sophomores, 25 juniors, and 25 seniors.

    Type of sample:

    Representative? Explain.

Go

Topic: Graphs of trigonometric functions

For each function identify the amplitude, period, horizontal shift, and vertical shift.

14. $f(t) = 120\cos\left(\frac{\pi}{4}(t-3)\right) + 30$

    Amplitude:

    Period:

    Horizontal Shift:

    Vertical Shift:

15. $f(t) = 3.5\sin\left(\frac{\pi}{6}t + \frac{1}{3}\right) + 7$

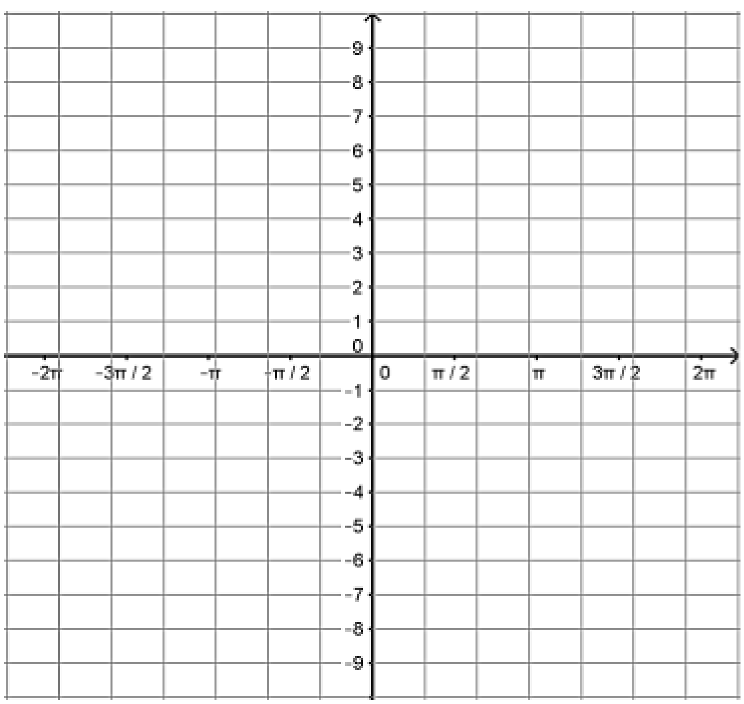
    Amplitude:

    Period:

    Horizontal Shift:

    Vertical Shift:

16. Graph $f(x) = \frac{1}{2}\sin(x-3) + 2$



Name **Statistics | 8.5**

Ready, Set, Go!

Ready
Topic: Two-way tables

The data below is the data from Mrs. Hender's class. Students needed to score a 60% or better to pass the test.

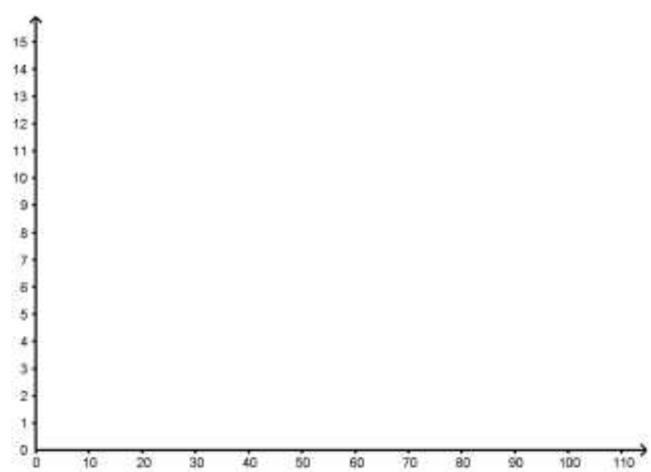
1st Period:	2nd Period:	3rd Period:
72, 83, 56, 63, 89, 92, 92, 67, 88, 84, 67, 97, 96, 100, 84, 82	80, 83, 81, 81, 67, 90, 70, 71, 72, 77, 81, 85, 86, 77, 74, 51	51, 45, 67, 83, 99, 100, 94, 52, 48, 46, 100, 59, 65, 56, 72, 63

1. Make a two-way frequency table showing how many students passed the test and how many failed each class.

	1st Period	2nd Period	3rd Period	Total
Passed				
Failed				
Total				

2. What percent of students passed Mrs. Hender's test in each class? What is the total percent that passed?

3. Use the data from all three classes to create a histogram. What properties of the normal curve does your histogram have?



4. If Mrs. Hender's were going to predict her total pass rate using only 2nd period, would she have a good prediction? Explain why or why not.

Set

Topic: Sampling methods

Determine which type of sampling method is represented in each scenario.

A. Simple random sample
B. Systematic random sample
C. Stratified random sample
D. Cluster random sample
E. Convenience sample
F. Volunteer sample

5. A researcher is studying the education levels of the population of Spain. The researcher randomly selects 20 cities in Spain and surveys every person in those cities.

6. Cardiff Kook Academy wants to examine the college and career goals of the students. Each student is assigned a random number. A random digit between 0 and 9 is selected to represent the first student to be surveyed and then every 20th student after that is surveyed.

7. Astroducks Coffee House wants to know how much caffeine college students drink in a day. Astroducks sends a surveyor out to the college that is closest to their headquarters in Carlsbad.

8. A group of 25 employees are chosen out of a hat from a company of 250 employees.

9. A company wants to know how much time the population spends answering email in a given day. They post a link on their website asking for people to take an online survey.

10. A company is investigating differences in income for various ethnicities. The company chooses to randomly survey 500 people in each of the designated ethnicities.

Go

Topic: Logarithms

Solve each equation below for *x* by applying properties for exponents and logarithms.

11. $2^{x-5} = 128$

12. $243^{x} = 27$

13. $3^{x+2} = 27^{x-3}$

14. $\log(2x+4) - \log(3x) = 0$

15. $\log_2(2x^2 + 4x - 2) - \log_2 10 = 0$

16. $\frac{\log 4x+2}{\log 15} = 1$

17. $\frac{\log_3(3x+6)}{\log_3 81} = 1$

Name **Statistics | 8.6**

Set, Go!

Set

Topic: Methods of investigating parameters of interest

For the following scenarios, identify each situation as a survey, observational study, or an experiment.

1. To determine if a new pain medication is effective, researchers randomly assign people into two groups: Group 1 receives pain medication and group 2 receives a placebo. Both groups are asked to rate their pain and the results are compared.

2. Officials want to determine if raising the speed limit from 75 mph to 80 mph will have an impact on safety. To determine this, they watch a stretch of the highway when the speed limit is 75 and see how many accidents there are. Then they observe the number of accidents over a period of time on the same stretch of highway for a speed limit of 80 mph. They then compare the difference.

3. To determine if a new sandwich on the menu is liked more than the original, the manager of the restaurant takes a random sample of customers that have tried both sandwiches and asks them which sandwich they prefer.

4. A newspaper wants to know what their customer satisfaction is. They randomly select 500 customers and ask them.

Mrs. Goodmore wants to know if doing homework actually helps students do better on their unit exams.

5. Describe how Mrs. Goodmore could carry out a survey to determine if homework actually helps. Explain the role of randomization in your design.

6. Describe how Mrs. Goodmore could carry out an observational study to determine if homework helps test scores.

7. Describe how Mrs. Goodmore could carry out an experiment to determine if homework helps test scores. Explain how you will use randomization in your design and how you will use a control.

8. If Mrs. Goodmore wants to determine if homework causes test scores to rise, which method would be best? Why?

Topic: Simulations

In 1963, NBC started to host a game called *Let's Make a Deal!* Contestants were given three doors to choose from. Behind one door was a prize. After selecting one door, the contestant was shown what was behind one of the doors they did not select. The contestant is then asked if they would like to stay with the door they first selected, or switch to the remaining one.

9. Which strategy do you think would result in the best chance of selecting the winning door? Should the contestant switch doors or stick with the first one they chose?

Go to the following website: http://www.shodor.org/interactivate/activities/SimpleMontyHall/. Select a door. One other door will be opened. Then select the same as the original door or switch doors and see if you win. Click "Let's do it again" to play another round.

10. Play the game 20 times using the "Stayed" method and 20 times using the "Switch" method. Record your wins and losses in the table below:

	Stick	Switch	Total
Win			
Lose			
Total			

11. Based on the simulation, what is $P(\text{winning}|\text{stay}) =$

12. Based on the simulation, what is $P(\text{winning}|\text{switch}) =$

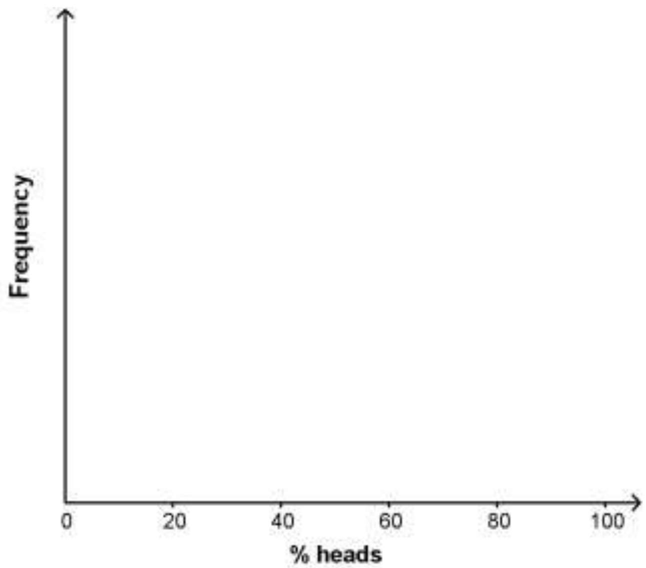
13. Repeat the process above to simulate 100 games for each strategy. What is the probability of winning using each method?

Topic: Features of histograms

14. Take a coin and flip it 5 times. Record the number of times the coin landed with heads up. Repeat this process 20 times either by hand or by simulation using technology, http://www.rossmanchance.com/applets/OneProp/OneProp.htm each time recording your results in the table below.

# Heads	**% Heads**	**Frequency**
0	0%	
1	20%	
2	40%	
3	60%	
4	80%	
5	100%	

15. Create a frequency bar graph of your results below. Describe the shape, center, and spread.

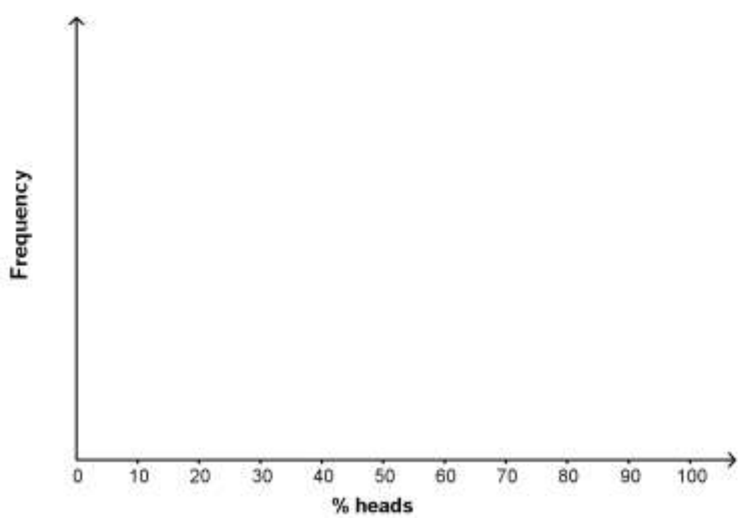


16. Flip a coin 20 times. Record the number of times heads lands up. Repeat this process 20 times either by hand or by simulation using technology.
    http://www.rossmanchance.com/applets/OneProp/OneProp.htm

    Record your results in the table below.

# Heads	% Heads	Frequency	# Heads	% Heads	Frequency
0	0%		11	55%	
1	5%		12	60%	
2	10%		13	65%	
3	15%		14	70%	
4	20%		15	75%	
5	25%		16	80%	
6	30%		17	85%	
7	35%		18	90%	
8	40%		19	95%	
9	45%		20	100%	
10	50%				

17. Create a frequency bar graph of your results below. Describe the shape, center, and spread.



18. Compare the shape, center, and spread of the graphs in questions 2 & 4. What do you notice?

19. If you repeated this process with 500 flips, instead of 5 or 20, predict what would happen to the shape, spread, and center of the new histogram.

Go

Topic: Normal curves

The average resting heart rate of a young adult is approximately 70 beats per minute with a standard deviation of 10 beats per minute. Assuming resting heart rate follows a normal distribution, answer the following questions.

20. Draw and label the normal curve that describes this distribution. Be sure to label the mean, and the measurements 1, 2, and 3 standard deviations away from the mean.

21. What percent of people have a heart rate between 55 and 80 beats per minute? Label these points on your normal curve in question 20 and shade in the area that represents the percent of people with heartbeats between 55 and 80 beats per minute.

22. If a resting heart rate above 80 beats per minute is considered unhealthy, what percent of people have an unhealthy heart rate?

Topic: z-scores

23. A normal distribution of scores has a standard deviation of 10. Find the z-scores corresponding to each of the following values:

    a. A score that is 20 points above the mean.

    b. A score that is 10 points below the mean.

    c. A score that is 15 points above the mean

    d. A score that is 30 points below the mean.

24. The Welcher Adult Intelligence Test Scale is composed of a number of subtests. On one subtest, the raw scores have a mean of 35 and a standard deviation of 6. Assuming these raw scores form a normal distribution:

    a. What number represents the 65^{th} percentile (what number separates the lower 65% of the distribution)?

    b. What number represents the 90^{th} percentile?

    c. What is the probability of getting a raw score between 28 and 38?

    d. What is the probability of getting a raw score between 41 and 44?

25. Scores on the SAT form a normal distribution with a mean of 500 and standard deviation of 100.

    a. What is the minimum score necessary to be in the top 15% of the SAT distribution?

    b. Find the z-scores that define the middle 80% of the distribution of SAT scores (372 and 628).

26. For a normal distribution, find the z-score that separates the distribution as follows:

    a. Separate the highest 30% from the rest of the distribution.

    b. Separate the lowest 40% from the rest of the distribution.

    c. Separate the highest 75% from the rest of the distribution.

Topic: Sampling methods

Which sampling method was utilized? Why?

27. Student organization looking to get signatures for a petition camp out in front of Class of 1950 Lecture Hall.

28. Select three students from a class to receive ice cream by putting all the students' names in a hat and picking out three names randomly.

29. Select three female students and three male students to receive ice cream by putting all the men's names in one hat and all the women's names in a different hat and picking out three names from each hat.

30. In Fall 1995, the BBC in Britain requested viewers to call the network and indicate their favorite poem.

31. Divide the class into four groups (freshman, sophomore, junior and senior) and take a random sample of two students from each group.

32. Priceline.com randomly e-mails a Customer Satisfaction Survey for certain transactions done on its site in which customers choose to either respond or not.