8.3 Relative Rates of Growth
QUESTION?????

Does $y = 2^x$ grow faster or slower than $y = x^2$?

How could you decide?

<table>
<thead>
<tr>
<th>X</th>
<th>Y₁</th>
<th>Y₂</th>
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<tbody>
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<td>9</td>
<td>512</td>
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</tr>
</tbody>
</table>

$X = 100$
The function $y = e^x$ grows very fast. We could graph it on the chalkboard: If $x$ is 3 inches, $y$ is about 20 inches:

![Graph showing exponential growth]

At 64 inches, the y-value would be at the edge of the known universe! (13 billion light-years)

At $x = 10$ inches, $y = \frac{1}{3}$ mile

At $x = 44$ inches, $y = 2$ million light-years

Let’s put rates of growth into perspective….
The function $y = \ln x$ grows **very** slowly. If we graph it on the chalkboard it looks like this:

By the time we reach the edge of the universe again (13 billion light-years) the chalk line will only have reached 64 inches!

We would have to move 2.6 **miles** to the right before the line moves a foot above the $x$-axis!

The function $y = \ln x$ increases everywhere, even though it increases extremely slowly.
Try This

For $y = \ln x$, what value of $x$ will give you $y \approx 13$?

$x \approx 445,000$

That’s really slooooww
Definitions: Faster, Slower, Same-rate Growth as \( x \to \infty \)

Let \( f(x) \) and \( g(x) \) be positive for \( x \) sufficiently large.

1. \( f \) grows faster than \( g \) (and \( g \) grows slower than \( f \)) as \( x \to \infty \) if

\[
\lim_{x \to \infty} \frac{f(x)}{g(x)} = \infty \quad \text{or} \quad \lim_{x \to \infty} \frac{g(x)}{f(x)} = 0
\]

2. \( f \) and \( g \) grow at the same rate as \( x \to \infty \) if

\[
\lim_{x \to \infty} \frac{f(x)}{g(x)} = L \neq 0
\]
WARNING

Please temporarily suspend your common sense.
According to this definition, $y = 2x$ does not grow faster than $y = x$. 

The book says that "$f$ grows faster than $g$" means that for large $x$ values, $g$ is negligible compared to $f$. 

"Grows faster" is not the same as "has a steeper slope"!

Since this is a finite non-zero limit, the functions grow at the same rate!
Which grows faster, $e^x$ or $x^2$?

\[
\lim_{x \to \infty} \frac{e^x}{x^2}
\]

\[
\lim_{x \to \infty} \frac{e^x}{2x}
\]

\[
\lim_{x \to \infty} \frac{e^x}{2} = \infty
\]

$e^x$ grows faster than $x^2$.

We can confirm this graphically:

\[y = \frac{e^x}{x^2}\]
“Growing at the same rate” is transitive.

In other words, if two functions grow at the same rate as a third function, then the first two functions grow at the same rate.
Example 4:

Show that \( f(x) = \sqrt{x^2 + 5} \) and \( g(x) = (2\sqrt{x} - 1)^2 \) grow at the same rate as \( x \to \infty \).

Let \( h(x) = x \)

\[
\lim_{x \to \infty} \frac{\sqrt{x^2 + 5}}{x} = \lim_{x \to \infty} \frac{\sqrt{x^2 + 5}}{\sqrt{x^2}} = \lim_{x \to \infty} \sqrt{\frac{x^2}{x^2} + \frac{5}{x^2}} = \lim_{x \to \infty} \sqrt{1 + \frac{5}{x^2}} = 1
\]

\[
\lim_{x \to \infty} \frac{(2\sqrt{x} - 1)^2}{x} = \lim_{x \to \infty} \frac{(2\sqrt{x} - 1)^2}{(\sqrt{x})^2} = \lim_{x \to \infty} \left( \frac{2\sqrt{x}}{\sqrt{x}} - \frac{1}{\sqrt{x}} \right)^2 = 4
\]

\[
\lim_{x \to \infty} \frac{f}{g} = \lim_{x \to \infty} \left( \frac{f}{h} \cdot \frac{h}{g} \right) = 1 \cdot \frac{1}{4} = \frac{1}{4}
\]

\( f \) and \( g \) grow at the same rate.
Definition  $f$ of Smaller Order than $g$

Let $f$ and $g$ be positive for $x$ sufficiently large. Then $f$ is of smaller order than $g$ as $x \to \infty$ if

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = 0$$

We write $f = o(g)$ and say “$f$ is little-oh of $g$.”

Saying $f = o(g)$ is another way to say that $f$ grows slower than $g$. 
Order and Oh-Notation

Definition \( f \) of at Most the Order of \( g \)

Let \( f \) and \( g \) be positive for \( x \) sufficiently large. Then \( f \) is of at most the order of \( g \) as \( x \to \infty \) if there is a positive integer \( M \) for which

\[
\frac{f(x)}{g(x)} \leq M \quad \text{for } x \text{ sufficiently large}
\]

We write \( f = O(g) \) and say “\( f \) is big-oh of \( g \).”

Saying \( f = O(g) \) is another way to say that \( f \) grows no faster than \( g \).