Exponential and Logarithmic Functions in Real Life

1. To see how quickly $f(x) = 2^x$ increases; let's perform the following thought experiment. Suppose we start with a piece of paper a thousandth of an inch thick, and we fold it in half fifty times. Each time we fold the paper, the thickness of the paper stack doubles, so the thickness of the resulting stack would be_____ inches. How thick do you think that is? It works out to be _____ miles!!!

2. **Sky Diving:** A sky diver jumps from a reasonable height above the ground. The air resistance she experiences is proportional to her velocity, and constant of proportionality is 0.2. It can be shown that the downward velocity of the sky diver at time t is given by $v(t) = 80\left(1 - e^{-0.2t}\right)$ where *t* is measured in seconds and *v(t)* is measured in feet per second (ft/s).

a) Find the initial velocity of the sky diver.
b) Find the velocity after 5 seconds and after 10 seconds.

3. **Compound Interest:** If $3000 is invested at an interest rate of 9% per year, find the amount of the investment at the end of 5 years for the following compounding methods.

(a) Annual
(b) Semiannual
(c) Monthly
(d) Weekly
(e) Daily
(f) Hourly
(g) Continuously

4. **Growth of an Exponential Function:** Suppose you are offered a job that lasts one month, and you are to be very well paid. Which of the following methods of payment is more profitable for you? Show your work.

(a) One million dollars at the end of the month.
(b) Two cents on the first day of the month, 4 cents on the second day, 8 cents on the third day, etc.

Logarithms

5. **Difficulty of a Task:** The difficulty in "acquiring a target" (such as using your mouse to click on an icon on your computer screen) depends on the distance to the target and the size of the target. According to Fitt's Law, the index of difficulty (ID) is given by $ID = \dfrac{\log\left(\dfrac{2A}{W}\right)}{\log 2}$ where *W* is the width of the target and *A* is the distance to the center of the target. Compare the difficulty of clicking on an icon that is 5mm wide to one that is 10mm wide. In each case, assume the mouse is 100mm from the icon.

6. **Wealth Distribution:** Vilfredo Pareto (1848-1923) observed that most of the wealth of a country is owned by a few members of the population. **Pareto's Principle** is $\log P = \log c - k \log W$ where *W* is the wealth level (how much money a person has) and *P* is the number of people in the population having that much money.

(a) Solve the equation for *P*.

(b) Assume $k = 2.1$, $c = 8000$, and *W* is measured in millions of dollars. Use part (a) to find the number of people who have $2 million or more. How many people have $10 million or more?

7. **Magnitude of Stars:** The magnitude *M* of a star is a measure of how bright a start appears to the human eye. It is defined by $M = -2.5\log\left(\dfrac{B}{B_o}\right)$ where B is the actual brightness of a star and B_o is a constant.

(a) Expand the right-hand side of the equation.

(b) Use part (a) to show that the brighter a star, the less its magnitude.

(c) Betelgeuse is about 100 times brighter than Albiero. Use part (a) to show that Betelgeuse is 5 magnitudes less than Albiero.

8. **True or False?** Discuss each equation and determine whether it is true for all possible values of the variables.

(a) $\log\left(\dfrac{x}{y}\right) = \dfrac{\log x}{\log y}$

(b) $\log_2(x - y) = \log_2 x - \log_2 y$

(c) $\log_5\left(\dfrac{a}{b^2}\right) = \log_5 a - 2\log_5 b$

(d) $(\ln P)(\ln Q) = \ln P + \ln Q$

(e) $\dfrac{\log a}{\log b} = \log a - \log b$

(f) $\log_a a^a = a$

(g) $\ln(x - y) = \dfrac{\ln x}{\ln y}$

(h) $-\ln\left(\dfrac{1}{A}\right) = \ln A$

9. **Compound Interest:** You want to invest $5000 in an account that pays 8.5% interest per year, compounded quarterly.

(a) Find the amount after 3 years

(b) How long will it take for the investment to double?

10. **Learning Curve:** A learning curve is a graph of a function P(t) that measures the performance of someone learning a skill as a function of the training time t. At first, the rate of learning is rapid. Then, as performance increases and approaches a maximal value M, the rate of learning decreases. It has been found that the function $P(t) = M - Ce^{-kt}$ where *k* and *C* are positive constants and $C<M$ is a reasonable model for learning.

(a) Express the learning time *t* as a function of the performance level *P*.

(b) For a pole-vaulter in training, the learning curve is given by $P(t) = 20 - 14e^{-0.024t}$ where *P(t)* is the height he is able to pole-vault after *t* months. After how many months of training is he able to vault 12 ft?

The next two problems use Newton's Law of Cooling.(D_o is the initial temperature difference between an object and its surroundings, and if its surroundings have a temperature T_s, then the temperature of the object at time t is modeled by the function $T(t) = T_s + D_o e^{-kt}$ where k is a positive constant that depends on the type of object)

11. **Cooling Soup:** A hot bowl of soup is served at a dinner party. It starts to cool according to Newton's Law of cooling so that its temperature at time *t* is given by $T(t) = 65 + 145e^{-0.05t}$ where *t* is measured in minutes and *T* is measured in °F.

(a) What is the initial temperature of the soup?
(b) What is the temperature after 10 minutes?
(c) After how long will the temperature be 100ºF?

12. **Time of Death:** Newton's Law of Cooling is used in homicide investigations to determine the time of death. The normal body temperature is 98°F. Immediately following death, the body begins to cool. It has been determined experimentally that the constant in Newton's Law of Cooling is approximately $k = 0.1947$, assuming time is measured in hours. Suppose that the temperature of the surroundings is 60°F.

(a) Find a function *T(t)* that models the temperature *t* hours after death.
(b) If the temperature of the body is now 72°F, how long ago was the time of death?

13. **The Richter Scale:** In 1935, the American geologist Charles Richter defined the magnitude M of an earthquake to be $M = \log \frac{I}{S}$ where *I* is the intensity of the earthquake (measured by the amplitude of a seismograph reading taken 100km from the epicenter of the earthquake) and *S* is the intensity of a "standard" earthquake (whose amplitude is 1 micron = 10^{-4}cm). Richter studied many earthquakes that occurred between 1900 and 1950. The largest had magnitude 8.9 on the Richter scale, and the smallest had a magnitude of 0. An earthquake of magnitude 6 is ten times stronger than an earthquake of magnitude 5. The 1906 earthquake in San Francisco had an estimated magnitude of 8.3 on the Richter scale. In the same year a powerful earthquake occurred on the Colombia-Ecuador border and was four times as intense.

(a) What was the magnitude of the Colombia-Ecuador earthquake on the Richter scale?
(b) In 1989 Loma Prieta earthquake that shook San Francisco had a magnitude of 7.1 on the Richter scale. How many times more intense was the 1906 earthquake than the 1989 quake?

14. **Agent 008:** Late last night, as Agent 008 negotiated his way back to headquarters after a long day, he saw the strangest glowing light. It came closer and closer until finally he could see what was creating the light. It was some kind of spaceship! He was frozen in his tracks. It landed only 15 feet from him, and a hatch slowly opened. Four little creatures came out carrying all sorts of equipment; including calculators and what appeared to be laser beams. They did not seem to notice that Agent 008 was there; that is, until he sneezed! Suddenly, the creatures turned around, looking very startled. They dashed into the spaceship, closed the hatch, and rocketed into the night. Could he believe what he had seen? Was it just a dream? After a few minutes of standing there dazed and confused, he started walking on, his eyes glazed over. He was just coming to his senses when he stepped on something strange. He picked it up and to his surprise it was one of the creature's calculators! What a prize! He started playing with it as he walked on. Boy! Headquarters is going to love this! It appeared to have a log button and as he played with it he noticed something interesting: log 10 did not equal 1 as it did on his calculator. With this calculator, $\log 10 \approx 0.926628408$! He tried some more: $\log 100 \approx 1.853256816$ and $\log 1000 \approx 2.77988524$. This was most peculiar. Obviously, the creatures did not work in base 10!

a) What base do the space creatures work in? Explain how you got your answer. You may want to rewrite the problem as $\log_b 10 = 0.926628408$ and try to figure out the value of b.

b) How many fingers do you think these space creatures have?

15. **Options:** Frances is just about to have his thirtieth birthday when he discovers he will inherit $100,000. Being a cautious young man, he wants to put it all in savings so he can retire on it when he is sixty-five. Unfortunately, Frances never took Math 3. He's afraid his investment advisor is lying to him when he tells him that there isn't that much difference between the three options (shown below), but option **C** is the best deal because it is compounded monthly. Frances figures the higher the interest rate, the better the deal. With clear algebra, show Frances which option he should take.

Option A.. . . . 7% annual interest rate, compounded annually.
Option B.. . . . 6.9% annual interest rate, compounded quarterly.
Option C.. . . . 6.8% annual interest rate, compounded monthly.

16. **Cup of Tea:** Aunt Eloise's house is always 20°C. She has just made a fresh cup of tea (tea is made with BOILING hot water—100°C). Five minutes after she made the tea her mad scientist nephew came in, stuck a thermometer in the cup and announced that the tea was now only 70°C. She had gotten involved with her book and forgot to have even a sip of her tea. Now she won't drink it because it isn't piping hot any more.

a) What three data points are given in the description above?
b) Write an equation that models this problem.
c) Use your equation to predict the temperature of the tea after 20 minutes.
d) Use your equation to predict when the tea will be body temperature—37°C.

17. **Money for College:** Mr. Anderson wants to make sure he has enough money for his daughter Samantha to go to college. When she was born, he put $5000 in the bank at 8.5% interest. How much money will she have on her 18th birthday if the interest is compounded **quarterly**? Little Samantha really wants to go to Cal Tech but needs $50,000 in the account before she can go. How old will she be before the account has that much money in this account?

18. **I Love Lucy:** Lucy and Ricky are working on a very long math assignment in which they are to solve all kinds of problems. In some problems they must just solve for x. In others, they must solve for x and y. After one hour of hard work and only getting half way through the assignment, Lucy is very frustrated.

"This is ridiculous!" Lucy exclaims. "I'm spending all my homework time on math. Doesn't our teacher realize we have other classes too!"
"Calm down, Lucy " Ricky said. "We only have 25 more problems to go. It should only take another couple of hours."
"Maybe for you, Ricky, but not for me!" With that, Lucy got up and got her graphing calculator.
"Lucy ! That's not going to help!" Ricky yelled.
"Sure it is" Lucy replied. "I'll whiz through the rest of these problems."
"No you won't! How do you intend to do this problem?"
With that, Ricky ripped off a piece of the homework assignment and shoved it into Lucy's face. The paper had this problem on it. $5(2.04)^x = 19$

Is Ricky right? Is Lucy doomed to slowly work out all of the problems? Or, is Lucy correct that she can use the graphing calculator to help solve the problem? What would be the most efficient way to solve this equation? Explain.

19. **Inheritance:** You have just inherited $10,000 from your incredibly old, rich (and generous) math teacher. However, there is one condition. You have to decide whether to put the money in a 9.5% annual percentage rate compounded yearly or in an 9.25% annual rate compounded quarterly. Each account will be compounded for 5 years. If you choose the account that generates the most interest, then you will receive the money in 5 years. Yeah, right, dream on!!

20. **You have Won Money:** Congratulations! You have just won Readers Digest Sweepstakes and you are waiting patiently for Ed McMahon to show up on your doorstep and present the check. There is one thing you have to decide before he gets here. You can either take the $50,000 in one lump sum or receive two installments at $25,000 each. The best interest rate available for the 50 grand is 4.75% per year compounded monthly, while for the two installments, the rate on the first 25 grand will be 5% per year and on the second installment (to be received one year later) will be 5% per year compounded quarterly.

a) Find the amount of money in your $50,000 account after 10 years.
b) Find the amount of money in your two $25,000 accounts after 10 years.
c) What do you notice about the effect of compounding interest?